the nmf weekly

Ask your math friend, James

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The Square Root of Two

G'Day!

This is your math friend James. Today I am answering a question from Ana Jean.

How do we know that the square root of two is irrational?

Ooh! This is a challenging question. There are a lot of complicated ideas to go through in very little space! This essay is going to make our brains hurt. Let's see how it goes.

First: What is the "square root" of a number?

I see two words in that term: one is square, and I know what that means, and the other is root, like the roots of a tree. In fact, roots are at the base of a tree and hold it in place, so I guess "square root" means: that which is at the base of a square and kinda holds it in place. Hmm!

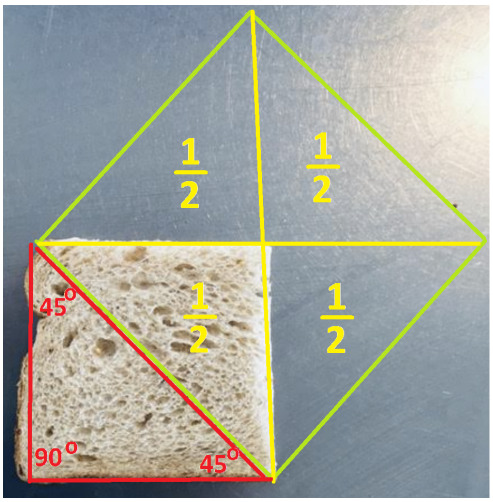
Here's a square of area 9. What "kinda holds it in place" as a square of area 9 is its base length, which is 3. And a square of area 25 has base length of 5.

We say that 3 is the square root of 9, and that 5 is the square root of 25. Also, 6 is the square root of 36, and 10 is the square root of 100. Square roots are all about the geometry of squares!

Second: What is the "square root of 2"?

Well, it is the base length of a square of area 2.

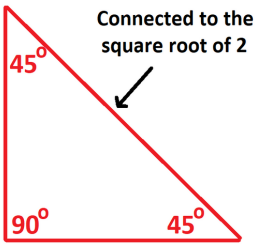
When I was making lunch I cut a piece of bread into a square of area 1. And then I realized, if I draw a diagonal line across my square piece, I could use it to draw a tilted square of area 2.



Do you see the green square in the picture has area 1/2 + 1/2 + 1/2 + 1/2 = 2?

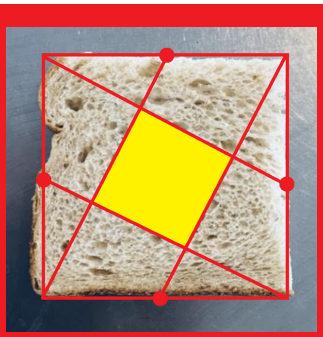
The diagonal across my square bread of area 1 is the base of a square of area 2 and so that diagonal has length the square root of 2!

If we have a bigger piece of square bread, we'd get a bigger multiple of the square root of 2. The diagonal of half a square is always connected with the square root of 2!



puzzle #1

In this picture, the square piece of bread has area 1.



Lines are drawn from each corner of the square to middle points of the sides of the square as shown.

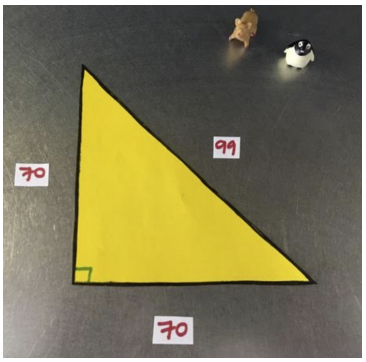
What is the area of the yellow square formed in the middle?

Third: What does it mean to say that the square root of 2 is "irrational"?

When people say ratio they often mean a comparison between two whole numbers. For example, people might say that "sugar to flour comes in 1 a to 3 ratio in this recipe." For something to be irrational means that it does not come as a ratio of whole numbers.

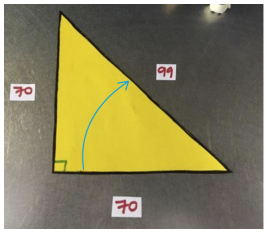
It turns out that the diagonal of half a square and its side length do not come in a whole number ratio. This makes people say "the square root of two is irrational."

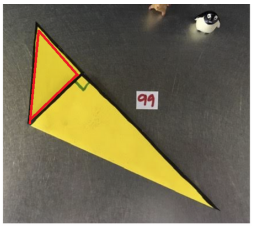
Here is a paper model of half a square and I am pretending that the lengths in this picture are whole numbers: 70, 70, and 99.

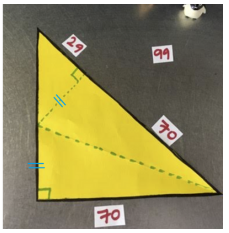


Something goes wrong with me pretending this.

To see what goes wrong, let's fold the triangle.

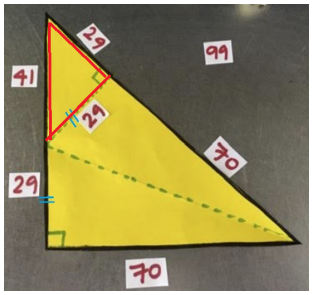






Warning: A REALLY REALLY BRAIN HURTY PART!

The side of length 70 is folded onto the side of length 99. We see another, smaller, half a square near the top. We can also see that one of it sides is 99-70 = 29.



In fact, if you mull on the picture for a while you will see that it also has a side of length 70 - 29 = 41.

So, now we have half a square with side lengths 29, 29, and 41. Do the folding trick again to get a half square with sides 12, 12, and 17. (Check this.)

In fact, we can do this over and over again to get a half square smaller than an atom, but still with whole number side lengths. This is impossible! This means we can't have a whole number ratio of sides to begin with. The square root of 2 has to be irrational. Whoa!

Check out MATHICAL for award-winning math books for middle-schoolers and teens, the YouTube channel NUMBERPHILE for math videos galore, and MORE MATH! for even more resources. Wowza!

Check out this NumberPhile video on the square root of 2.

puzzle #2

This time I've drawn lines across my square piece of bread to thirdway points along its sides.

What is the area of one of the yellow squares formed in the middle?

Do you have a math question for me to answer, or try to answer?

Write to me at the website. Each week I'll pick a new question and give my thoughts on it!

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The NMF Weekly is written by mathematician Dr. James Tanton as a resource for friends and fans of the 2021 National Math Festival.

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