Exploding Dots™ Guide

Experience 7:

Infinite Sums

Here we play with "infinitely long polynomials" and discover the geometric series formula, and more.

Year levels: High School and up

Access videos of *Exploding Dots™* lessons at www.globalmathproject.org.
Printable student handouts for this experience are available separately.

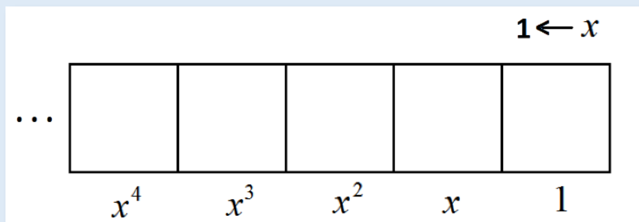
Experience 7: Infinite Sums
Overview

Objectives

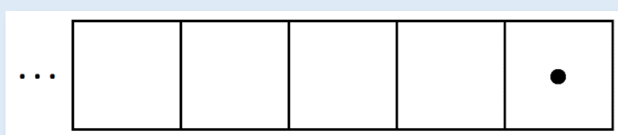
Let's conduct polynomial division to discover infinite sums! (And then let's contemplate the meaning of these sums.)

The Experience in a Nutshell

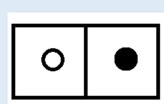
Consider again the $1 \leftarrow x$ machine.



Let's use this machine compute this strange division problem: $\frac{1}{1-x}$. It is the very simple polynomial 1, which looks like this



divided by $1-x$, which looks like this, one antidot and one dot.



We'll see the answer following appear.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

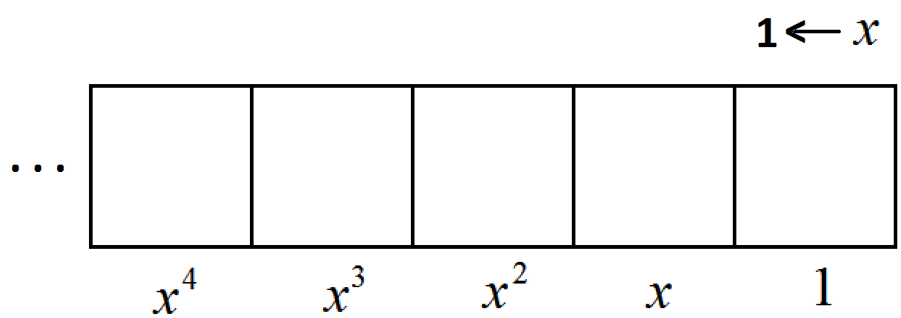
And this is the famous *geometric series formula.*

Infinite Sums

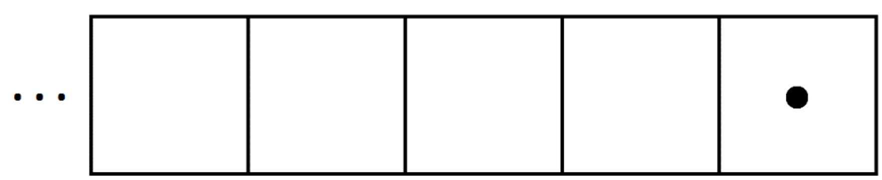
In the previous Experience we played with the $1 \leftarrow x$ machine and saw the power of that machine to make advanced school algebra so natural and straightforward.

In this Experience, let's take that power to infinity! And will be equally natural and straightforward.

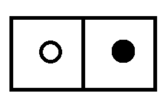
Consider again the $1 \leftarrow x$ machine.



And let's use this machine compute this strange division problem: $\frac{1}{1-x}$. This is the very simple polynomial 1, which looks like this



divided by $1-x$, which looks like this, one antidot and one dot.



Do you see any one-antidot-and one-dot pairs in the picture of just 1? Nope!

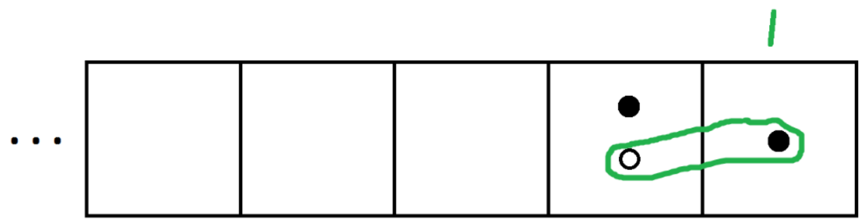
But remember

If there is something in life you want, make it happen! (And deal with the consequences.)

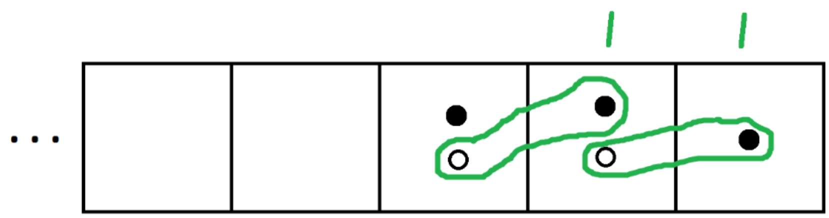
Can we make antidiot-dot pairs appear in the picture? Wouldn't it be nice to have an antidot to the left of the one dot we have?

Well, make it happen!

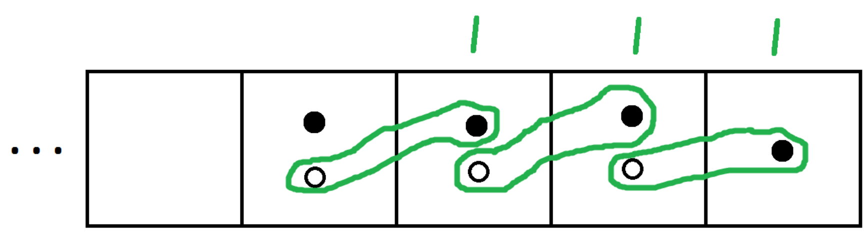
And to keep that box technically empty we need to add a dot as well. That gives us one copy of what we want.



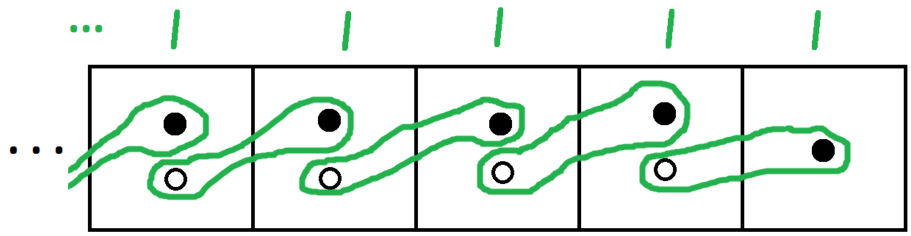
And we can do it again.



And again.



In fact, we can see we'll be doing this forever!



Whoa!

How do we read this answer?

Well, we have one 1, and one x, and one x^2, and one x^3, and so on. We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots.$$

The answer is an infinite sum.

The equation we obtained is a very famous formula in mathematics. It is called the *geometric series formula* and it is often given in many upper-level high school text books for students to use. But textbooks often write the formula the other way round, with the letter r rather than the letter x.

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

In a calculus class, one might say we've just calculated the Taylor series of the rational function $\frac{1}{1-x}$. That sounds scary! But the work we did with dots-and-boxes shows that is not at all scary. In fact, it all kind of fun!

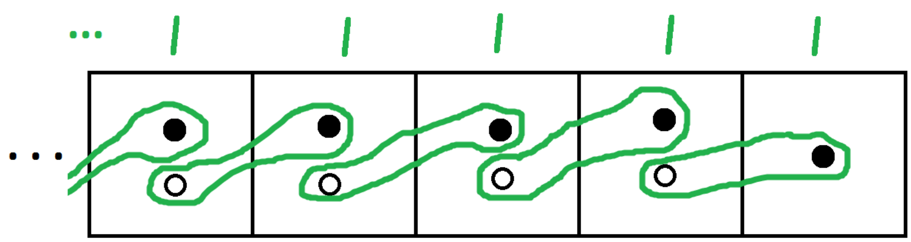
Exploding Dots

Experience 7: Infinite Sums

Access videos of all *Exploding Dots* lessons at www.globalmathproject.org.

Practice Set A: Infinite Sums

This picture shows that 1 divided by $1-x$ is $1+x+x^2+x^3+\cdots$. (Many dot-antidot pairs were inserted along the way!)



Here are some questions for you to try if you want.

1. *Use dots-and-boxes to show that* $\frac{1}{1+x}$ *equals* $1-x+x^2-x^3+x^4-\cdots$.

2. *Compute* $\frac{x}{1-x^2}$. *Do you get a sum of odd powers of* x *?*

This next question is really cool! I advise you to draw very big boxes when you draw your dots and boxes picture. (The number of dots you need grows large quite quickly.)

3. *Compute* $\frac{1}{1-x-x^2}$ *and discover the famous Fibonacci sequence!*

Solutions to Practice Set A

1. Do it!

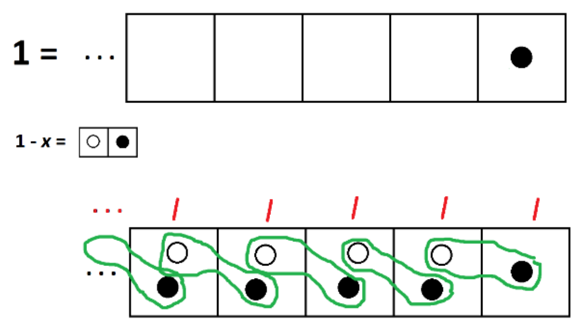
2. You get $x + x^3 + x^5 + x^7 + \cdots$, which makes sense as you can think of the expression as $x \times \frac{1}{1-(x^2)}$ and this is $x \times \left(1 + (x^2) + (x^2)^2 + (x^2)^3 + \cdots\right)$.

3. You get $\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \cdots$.

Optional: Should We Believe Infinite Sums?

The geometric series formula is $1+x+x^2+x^3+\cdots=\frac{1}{1-x}$.

We saw this by taking the polynomial 1 and dividing it by the polynomial $1-x$. The infinite sum of the powers of x naturally appear.



But can this formula be true?

What if x is, say, the number 2 ? Then the geometric series formula says that $1+2+4+8+16+\cdots$ equals $\frac{1}{1-2}$, which is -1. That's just absurd!

On the other hand, if $x=0.1$, then the geometric series formula says that

$$\begin{aligned}1+(0.1)+(0.1)^2+(0.1)^3+\cdots &= 1+0.1+0.01+0.001+\cdots \\ &= 1.111...\end{aligned}$$

equals

$$\frac{1}{1-0.1}=\frac{1}{0.9}=\frac{10}{9}.$$

This is one and one-ninth. From primary school we learn that $\frac{1}{9}$ as a decimal is $0.111...$ (or look at Experience 8). So one-and-one-ninth equals $1.111...$. The geometric series formula is correct in this case!

$$1+(0.1)+(0.1)^2+(0.1)^3+\cdots=\frac{1}{1-0.1}.\quad \text{YES!}$$

What is going on? When can we believe the formula and when can we not?

ALGEBRA VERSUS ARITHMETIC

The $1 \leftarrow x$ machine is a machine that displays mechanical algebra. Everything it does and all we conclude from it is true in terms of mechanical algebra.

For example, the geometric series formula says that $1+2+4+8+\cdots$ equals $\frac{1}{1-2}$, which is absurd as a statement of arithmetic. However, in a purely mechanical sense, without regard to arithmetic, there is some version of truth to this claim. Here's how we can see some truth.

> If $1+2+4+8+\cdots$ equals $\frac{1}{1-2}$, then multiplying $1+2+4+8+\cdots$ by $1-2$ should give 1. Does it?
>
> Yes it does!
>
> $$\begin{aligned}(1-2)\times(1+2+4+8+\cdots) &= (1-2)+(1-2)\times 2+(1-2)\times 4+(1-2)\times 8+\cdots \\ &= 1-2+2-4+4-8+8-16+\cdots \\ &= 1+0+0+0+\cdots \\ &= 1.\end{aligned}$$
>
> So $1+2+4+8+\cdots$ really does behave like $\frac{1}{1-2}$.

But still, this is not very satisfying. We want to know when $1+x+x^2+x^3+\cdots=\frac{1}{1-x}$ is actually true as a statement of arithmetic.

THE CALCULUS ANSWER (FOR THE BOLD)

One studies infinite sums in a calculus class. You learn there, for example, that $1+x+x^2+x^3+\cdots=\frac{1}{1-x}$ is true as a statement of arithmetic for small values of x (specifically, for all values of x strictly between -1 and 1.) The formula is valid for $x=0.1$, as we saw, and not for $x=2$.

For those who are game, here is how the calculus argument goes.

Regular polynomial division shows that $\frac{1-x^2}{1-x}=1+x$ and $\frac{1-x^3}{1-x}=1+x+x^2$ and $\frac{1-x^4}{1-x}=1+x+x^2+x^3$, and so on. (Try these!) In general, we see that

$$1+x+x^2+\cdots+x^{n-1}=\frac{1-x^n}{1-x}.$$

Now as we let n get bigger and bigger it looks like we're getting the infinite geometric sum.

$$1+x = \frac{1-x^2}{1-x}$$

$$1+x+x^2 = \frac{1-x^3}{1-x}$$

$$1+x+x^2+x^3 = \frac{1-x^4}{1-x}$$

$$\downarrow \qquad\qquad \downarrow$$

$$1+x+x^2+x^3+\cdots = \textbf{?}$$

So the question is: What does $\frac{1-x^n}{1-x}$ go to as n gets bigger and bigger? If there is an answer to this question, then the answer will be the value of $1+x+x^2+x^3+\cdots$.

So does $\frac{1-x^n}{1-x}$ have a limit value? Well, this depends on whether or not x^n has a limit value as n grows. So for which values of x do the powers of it approach a value?

We know the that powers of 0.1, for example, and of 0.83 and of $-\frac{1}{2}$, all approach zero for bigger and bigger powers. In fact, x^n gets closer and closer to zero as n grows for any value of x between -1 and 1.

So for $-1<x<1$, we have

$$\begin{array}{ccc}
1+x & = & \dfrac{1-x^2}{1-x} \\
1+x+x^2 & = & \dfrac{1-x^3}{1-x} \\
1+x+x^2+x^3 & = & \dfrac{1-x^4}{1-x} \\
\downarrow & & \downarrow \\
1+x+x^2+x^3+\cdots & = & \dfrac{1-0}{1-x}=\dfrac{1}{1-x}
\end{array}$$

The geometric series formula can be believed, as a statement of arithmetic, for $-1<x<1$, at least.

THE HONEST APPROACH

Another approach to examining the geometric series formula is to make an explicit leap of faith. Let's assume that the infinite sum $1+x+x^2+x^3+\cdots$ is meaningful and has an answer. Call that answer S. Then

$$\begin{aligned} S &= 1+x+x^2+x^3+\cdots \\ &= 1+x\left(1+x+x^2+\cdots\right) \\ &= 1+xS \end{aligned}$$

from which we get $S=\frac{1}{1-x}$.

This is an honest approach. It proves: *IF the infinite sum* $1+x+x^2+\cdots$ *has an answer, then that answer must be* $\frac{1}{1-x}$. It makes no assertion as to whether or not the infinite sum is meaningful and has an answer in the first place.

The same is true for the dots-and-boxes approach as it too proves: IF $1+x+x^2+x^3+\cdots$ is meaningful to you, then it has the answer $\frac{1}{1-x}$. It is up to you to decide if this infinite sum is meaningful. (Calculus likes to say it is if $-1<x<1$.)

OTHER SYSTEMS OF ARITHMETIC OFFER OTHER MEANINGS

The statement $1+2+4+8+\cdots=-1$ is meaningless in ordinary arithmetic. But who says we have to stay with ordinary arithmetic? Is there an extraordinary way to view matters?

We tend to view numbers as spaced apart on the number line additively. Walk one step to the right of 0 and we end up at position 1. Now add to that two steps and we end up position 3. Now add four steps, position 7. And so on. The sum $1+2+4+8+\cdots$, in this viewpoint, takes us infinitely far to the right of 0 on the number line.

$1+2+4+8+\cdots=\infty$ in ordinary arithmetic. It does not equal -1.

But let's think of numbers multiplicatively. In particular, since we are focusing on the sum $1+2+4+8+\cdots$ let's think of factors and multiples of powers of two.

Now 0 is a highly divisible number. It is the most divisible number of all. With regard to just two-ness it can be divided by 2 once, in fact twice, in fact thrice. In fact, you can divide 0 by two as many times as you like—and still keep dividing.

With regard to two-ness, the number 8 is somewhat zero-like: you can divide it by two three times. But 32 is even more zero-like: you can divide it by two five times. And 2^{100} is even more zero-like still.

So in this sense, 2^{100} is a number very close to 0. The number 32 is somewhat close to 0. The number 8 is less close. The number 1 is not very close to zero at all: it cannot be evenly divided by two even once.

So, in this context, could $1+2+4+8+\cdots$ possibly be -1?

Well

$$
\begin{aligned}
&1+2=3 && =4-1 \\
&1+2+4=7 && =8-1 \\
&1+2+4+8=15 && =16-1 \\
&\vdots \\
&1+2+\cdots+2^{99} && =2^{100}-1
\end{aligned}
$$

These finite sums grow to become "a number very close to zero, minus one." In the limit, the infinite sum thus has value $0-1=-1$, just as our formal arguments said it would be.

So in this multiplicative view of arithmetic, $1+2+4+8+\cdots$ is a meaningful quantity, and it does indeed have value -1. The geometric series formula is meaningful and correct for $x=2$ in this context.

The point is that our dots-and-boxes work shows what the answers to many infinite sums have to be *IF* the infinite sum has meaning to you. It is thus up to you to decide what arithmetic context you want to play in and whether or not the infinite sum you are examining is meaningful in that context. If it is, then dots and boxes tells you its answer!

Go to the chapter *Unusual Mathematics for Unusual Numbers* if you are intrigued by the ideas presented here.