



# *Exploding Dots™ Guide*

## Experience 12:

### PUZZLES explained by EXPLODING DOTS

*Here is a sample of puzzles—some classic, some new—that can be beautifully analyzed and well understood with an Exploding Dots approach. Each piece begins with a discussion on introducing the puzzle to students along with some possible thoughts, questions, and reactions students might have to the puzzles.*

*The suggested age levels are very loose! Many elementary grade students will enjoy a number of these puzzles and any puzzle labeled for a certain grade level, when probed more deeply, can unfold to more challenging queries. Do indeed look at the suggested puzzle extensions.*

*The solutions and suggested fun extensions to some of the puzzles are presented in the second half of this document. We give away all the content for the remaining puzzles as a gift for signing up to our newsletter.*

*These puzzle essays, and more as they are developed, appear on the GLOBAL MATH PROJECT website.*

*We also thank out partner organization BRILLIANT for sharing some of their puzzles with us.*



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*Enjoy!*

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## THE FIVE-CARD MIND-READING TRICK



This is a classic number trick that serves as a perfect start to the entire EXPLODING DOTS story. Students just love this “mind-reading” trick and are fascinated by it. As soon as one’s understanding of  $1 \leftarrow 2$  machine codes come to light, the magic of this trick suddenly crystalizes. It’s an AHA moment!

**EXPLODING DOTS Topic:**

**Experiences 1 and 2:** Understanding  $1 \leftarrow 2$  machine (binary) codes.

**Suggested Grade Level:**

Middle School, High School, and All.

*Our partner organization BRILLIANT also has plenty of materials that tie in beautifully with this topic. See the **Further Reading** at the end of the piece.*



## THE FIVE-CARD MIND-READING TRICK



Here's a classic trick. On a board write out the following five groups of numbers.

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

Now ask your students to each silently think of a number between 1 and 31. (They may choose the number 1 itself if they like or the number 31.)

Now perform the mindreading trick by having the following conversation with a student.

"Suzzy. Is the number you are thinking of in group A?"	"Yes."
"Is the number you are thinking of in group B?"	"Yes."
"Is it in group C?"	"No."
"Is it in group D?"	"No."
"Group E?"	"Yes!"

"Ahh ... your number is 25."

"Wow! Yes it is!"

Have the same conversation with a few more students, noting each time which groups elicit a "yes" answer from a given student. The secret number they have in mind is simply the sum of the top-left corner numbers in each group with a yes answer. For example, Suzzy answered YES YES NO NO YES. Groups A, B, and E have top left numbers 16, 8, and 1, respectively, and indeed  $16 + 8 + 1 = 25$ .

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
<b>16</b> 20 24 28	<b>8</b> 12 24 28	4 12 20 28	2 10 18 26	<b>1</b> 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31
<b>YES</b>	<b>YES</b>	<b>NO</b>	<b>NO</b>	<b>YES</b>

*Suzzy's responses for the number 25.*

As practice, check that if Sameer is thinking the number 13, he will answer NO YES YES NO YES and indeed  $8 + 4 + 1 = 13$ .

Do this as many times as your students desire. Invite them to figure out what you are doing, and then, why what you are doing works! (Can you figure out why the top left numbers are key?) Perhaps facilitate thinking by writing a student's YES/NO answers under each group each time as done for Suzzy above.

## Some Things Students Might Notice or Question

1. Students might notice that group E contains all the odd numbers.
2. Students might notice that group A contains all the numbers 16 and above.
3. Students might question the number 31. “Why must we choose between 1 and 31. What’s special about the number 31?”
4. Students might notice the power of two in the top left corner of each group. (Though you might prefer to jumble the numbers in each group when you write them on the board to make this trick more mysterious!)

## CRYPTOGRAM from BRILLIANT



Our partner BRILLIANT ([www.brilliant.org](http://www.brilliant.org)) has a tremendous series of cryptogram puzzles—as part of an astonishing library of general puzzles and innovative mathematics content—that help students develop a profound and strong understanding of the standard arithmetic algorithms. Here are two puzzles from their *Practice Contest Math* series at <https://brilliant.org/practice/whats-the-number/>.



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**EXPLODING DOTS Topic:**

**Experience 3:** Addition in a  $1 \leftarrow 10$  machine.

**Suggested Grade Level:**

Middle School, High School, and All.

## CRYPTOGRAM from BRILLIANT



A cryptogram is a piece of arithmetic using standard algorithms—standard long addition or standard long multiplication, for instance—with some digits replaced by letters. If a letter is repeated, it means that the same digit appears in those places. Different letters represent different digits and no letter represents a digit you see unchanged.

When your students see this first example, they will likely immediately understand the challenge.

**Puzzle 1** from <https://brilliant.org/practice/whats-the-number/>

$$\begin{array}{r} \phantom{+} \phantom{1} A \phantom{2} A \\ + \phantom{+} 1 \phantom{A} A \\ \hline \phantom{+} 8 \phantom{A} 2 \end{array}$$

What digit in place of  $A$  would make this sum true?

Your students might simply start guess as to which digit  $A$  could be. It is clearly not  $0$ , leaving it as either  $3, 4, 5, 6, 7,$  or  $9$ , the only digits not showing.

Of course, looking at the units column, it does first appear that  $A$  should be  $1$ . But then one quickly realizes that the tens column is troubled. Allow your students to think through this, if they seem to be going that route. But do remind them afterwards that one of the features of the puzzle is that letters represent digits that aren't already showing.

Students will likely quickly realize that the ones column must add to  $12$  instead of  $2$ , and this happens only for  $A = 6$ . All then does work out beautifully for the tens column too.

After solving this puzzle, suggest to students that they could alternatively think though this puzzle via the setup of a  $1 \leftarrow 10$  machine.

Can they—and you—make sense of this rephrasing of the puzzle?

$$\begin{array}{|c|c|} \hline A+1 & 2A \\ \hline \end{array} \stackrel{1 \leftarrow 10}{=} \begin{array}{|c|c|} \hline 8 & 2 \\ \hline \end{array}$$

We can see right away that  $A = 1$  does not work.

Let's play with explosions and unexplosions in the  $1 \leftarrow 10$  machine and rewrite the right side.

$$\begin{array}{l}
 \begin{array}{c} 1 \leftarrow 10 \\ \boxed{A+1} \quad \boxed{2A} \end{array} = \begin{array}{|c|c|} \hline 8 & 2 \\ \hline \end{array} \\
 = \begin{array}{|c|c|} \hline 7 & 12 \\ \hline \end{array} \leftarrow \\
 = \begin{array}{|c|c|} \hline 6 & 22 \\ \hline \end{array} \\
 \vdots
 \end{array}$$

In fact, after just one explosion, we see that the expression  $7|12$  matches the left with  $A = 6$ .

Ask your students if they might like to try EXPLODING DOTS thinking on this next challenging question.

Puzzle 2 from <https://brilliant.org/practice/whats-the-number/>

$$\begin{array}{r}
 A \quad B \quad C \\
 \quad B \quad C \\
 + \quad \quad C \\
 \hline
 C \quad 7 \quad B
 \end{array}$$

What is the value of the three-digit number  $\overline{ABC}$ ?

## Some Things Students Might Notice or Question

5. Your students will likely present the problem in a  $1 \leftarrow 10$  machine as follows.

$$\begin{array}{|c|c|c|} \hline & & 1 \leftarrow 10 \\ \hline \color{red}{A} & \color{blue}{2B} & \color{green}{3C} \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \color{green}{C} & \color{black}{7} & \color{blue}{B} \\ \hline \end{array}$$

6. The entry  $2B$  in the tens place on the left cannot directly match the entry  $7$  in the tens place on the right.
7. Comparing leftmost boxes, maybe  $A = C$ ? But since they are meant to be distinct digits, this can't be the case. But how could it be otherwise?
8. Is  $3C = B$ ?
9. Maybe we can unexplode one of the tens on the right to look at this picture instead?

$$\begin{array}{|c|c|c|} \hline & & 1 \leftarrow 10 \\ \hline \color{red}{A} & \color{blue}{2B} & \color{green}{3C} \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \color{green}{C} & \color{black}{6} & \color{blue}{B+10} \\ \hline \end{array}$$

Could  $B = 3$ ?

10. This is hard!

## MULTIPLICATION PUZZLE from BRILLIANT



Our partner BRILLIANT ([www.brilliant.org](http://www.brilliant.org)) has an astonishing library of general puzzles and innovative mathematics content. We start here with a warmup puzzle on long multiplication from Brilliant, and then push it further with the power of the *Exploding Dots* mindset! Our starting puzzle is item 2 from <https://brilliant.org/practice/calculation-tricks/?p=2>.



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**EXPLODING DOTS Topic:**

**Experience 3:** Combining long multiplication with *Exploding Dots* thinking.

**Suggested Grade Level:**

Middle School, High School, and All.

## MULTIPLICATION PUZZLE from BRILLIANT



Start by sharing this challenge from Brilliant.org with your students and asking if they can guess what the answer is going to be. Then have them confirm their guess by actually performing the long multiplication by hand.

Puzzle 2 from <https://brilliant.org/practice/calculation-tricks/?p=2>.

$$1 \times 1 = 1$$

$$11 \times 11 = 121$$

$$111 \times 111 = 12321$$

$$1111 \times 1111 = 1234321$$

Evaluate

$$11111 \times 11111.$$

We see that if the number that is a string of five 1s is multiplied by itself we get the answer 123454321.

$$\begin{array}{r}
 11111 \\
 \times 11111 \\
 \hline
 11111 \\
 111110 \\
 1111100 \\
 11111000 \\
 111110000 \\
 \hline
 = 123454321
 \end{array}$$

Ask your students:

*What answer do you expect if the number that is a string of eight 1s is multiplied by itself?*

$$11111111 \times 11111111 = ?$$

They will likely guess the answer 123456787654321 and can check this if they like by writing the long multiplication algorithm.

Now comes today's puzzle.

The number that is a string of ten 1s is multiplied by itself. What answer do we obtain?

$$1111111111 \times 1111111111 = ?$$

The number that is a sting of fourteen 1s is multiplied by itself. What answer do we obtain?

$$11111111111111 \times 11111111111111 = ?$$

### Some Things Students Might Notice or Question

1. Some students might wish to write out the long multiplications. That's okay.
2. Students will likely comment that for the first product the middle digit needs to be 10, and 14 for the second product.
3. Students who have followed *Exploding Dots* might suggest we can perform explosions on these digits.

## A TWO-BUTTON CALCULATOR



This puzzle asks students to exercise the power of place-value and the effect of multiplying a number by the value of the base in which it is presented. (Multiplying a number written in base ten, for instance, by 10 has the apparent effect of appending a 0 to the number.)

**EXPLODING DOTS Topic:**

**Experience 2:** Understanding the place-value machines.

**Experience 4:** Multiplying by the base number in a machine.

**Suggested Grade Level:**

Middle School, High School, and All.

## A TWO-BUTTON CALCULATOR



Present to your students the following three puzzles, either one at a time or altogether at the same time.

**Puzzle 1:** A calculator, currently displaying  $0$ , has just two buttons.



Pressing this button adds 1 to the number currently on display.



Pressing this button multiplies the number currently on display by ten.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number  $5304$  appear on screen?

If so, what is the least number of button presses needed to do it?

**Puzzle 2:** A calculator, currently displaying  $0$ , has just two buttons.



Pressing this button adds 1 to the number currently on display.



Pressing this button doubles the number currently on display.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number  $244$  appear on screen?

If so, what is the least number of button presses you needed to do it?

**Puzzle 3:** A calculator, currently displaying  $0$ , has just two buttons.



Pressing this button adds 1 to the number currently on display.



Pressing this button triples the number currently on display.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number  $244$  appear on screen?

What is the least number of button presses needed to do it?

## Some Things Students Might Notice or Ask

1. You can always just repeatedly press the  $+$  button (and never press the  $\times$  button). That way you can get any number to appear on display. But it will take 5034 presses to make 5304 to appear in the first puzzle, and 244 presses in each of the remaining two puzzles to make 244 appear.
2. Pressing the  $\times$  first never does you any good: multiplying zero by ten (or by two or by three) keeps you at zero.
3. You can get to 5000, at least, in the first puzzle by pressing  $+$  five times and then  $\times$  three times.
4. You can get to 5304 in the first puzzle by

Pressing  $+$  five times.  
Pressing  $\times$ .  
Pressing  $+$  three times.  
Pressing  $\times$ .  
Pressing  $\times$ .  
Pressing  $+$  four times.

That's fifteen button presses. Can we do it in less?

5. Puzzle 1 is hard enough!
6. In puzzle 1, would it ever be efficient to press the  $+$  button ten times in a row?

## PEASANT MULTIPLICATION



Sometimes called “Russian Multiplication” or even “Russian Peasant Multiplication,” the curious multiplication technique we discuss here has close connections to an ancient multiplication technique employed more than three-and-a-half millennia ago in Egypt. That older technique is described in the famous Rhind Papyrus and is sometimes called “Egyptian Multiplication” or even “Ethiopian Multiplication,” and we’ll describe it too!

**EXPLODING DOTS Topic:**

**Experience 2:** Understanding the  $1 \leftarrow 2$  machine.

**Suggested Grade Level:**

Middle School, High School, and All.

## PEASANT MULTIPLICATION



Demonstrate the following multiplication technique, often called “Peasant Multiplication,” on the classroom board.

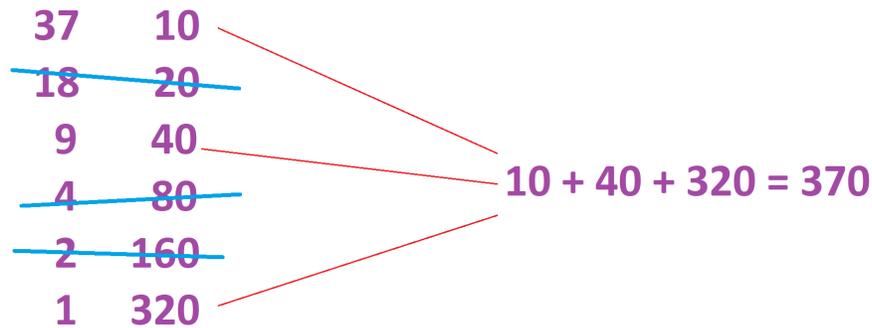
Here’s a curious way to perform long multiplication.

To illustrate the technique, let’s compute  $37 \times 10$ , whose answer we know.

- Start by heading two columns with the two numbers in the product.
- Repeatedly halve the numbers in the left column and double the numbers in the right – and to make life easy, ignore any fractions that arise. Stop when you see 1 on the left.

37	10	
<del>18</del>	<del>20</del>	
9	40	
<del>4</del>	<del>80</del>	
<del>2</del>	<del>160</del>	
1	320	

$10 + 40 + 320 = 370$



- Delete any row that has an even number on the left and sum the numbers that survive on the right.
- The sum of the surviving numbers is the desired product!

**Practice:** Use this technique to show that  $10 \times 37$  also gives the answer 370.

Have your students practice this technique with some more examples.

Of course, today’s question is: *Why does this technique work?*

**A Nudge:** You could perhaps ask your students to examine the products  $8 \times N$  and  $64 \times N$  via this technique where  $N$  represents some general (unspecified) number.

### Some Things Students Might Notice or Ask

1. This is weird!
2. Is it really okay to ignore remainders?
3. Working out  $10 \times 37$  is awkward.
4. You can work out  $10 \times 37$  in an Exploding Dots way.

$$\begin{array}{r}
 \cancel{10} \quad \cancel{37} \\
 5 \quad 6|14 \\
 \cancel{2} \quad \cancel{12|28} \\
 1 \quad 24|56
 \end{array}
 \quad \rightarrow \quad
 6|14 + 24|56 = 30|70 = 370$$

5. To work out  $8 \times N$  we end up doubling  $N$  three times to get  $8N$ , and only this survives on the right side. To work out  $64 \times N$  we end up double  $N$  six times to get  $64N$ , and only this survives on the right side.

$$\begin{array}{r}
 \cancel{8} \quad \cancel{N} \\
 \cancel{4} \quad \cancel{2N} \\
 \cancel{2} \quad \cancel{4N} \\
 1 \quad 8N
 \end{array}$$

6. You don't actually have to stop at 1 on the left side: you can keep going to get zeros. But since zero is an even number, these rows get crossed out in any case.

$$\begin{array}{r}
 37 \quad 10 \\
 \cancel{18} \quad \cancel{20} \\
 9 \quad 40 \\
 \cancel{4} \quad \cancel{80} \\
 \cancel{2} \quad \cancel{160} \\
 1 \quad 320 \\
 \cancel{0} \quad \cancel{640} \\
 \cancel{0} \quad \cancel{1280} \\
 \cancel{0} \quad \cancel{2560} \\
 \vdots
 \end{array}
 \quad \rightarrow \quad
 10 + 40 + 320 = 370$$

## A SUM OF FRACTIONS from BRILLIANT



Our partner BRILLIANT ([www.brilliant.org](http://www.brilliant.org)) has an astonishing library of general puzzles and innovative mathematics content. Here's a clever puzzle from Brilliant on converting fractions into decimals and making use of patterns. It's item 4 from this Practice Set: <https://brilliant.org/practice/problem-solving-4/?chapter=divisibility-in-other-bases>.



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**EXPLODING DOTS Topic:**

**Experience 8:** Decimals in base 10.

**Suggested Grade Level:**

Middle School, High School, and All.

## A SUM OF FRACTIONS from BRILLIANT



Have your students compute  $1 \div 9$  in a  $1 \leftarrow 10$  machine and thus rewrite the fraction  $\frac{1}{9}$  as an infinite decimal. That might well be enough of a warm-up for this curious challenge from BRILLIANT.

**Puzzle 4** from <https://brilliant.org/practice/problem-solving-4/?chapter=divisibility-in-other-bases>.

$$S = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999}$$

If  $S$  is evaluated as a decimal, what is the 24<sup>th</sup> digit to the right of the decimal point?

### Some Things Students Might Notice or Question

- Can we just type this into a calculator?

The answer is, of course, YES! Have students try it. Does it prove to be helpful?

- Should we work out the decimal expansions of the remaining four fractions in the sum?
- Is there something special about the 24<sup>th</sup> decimal place?
- Why do we care?

## DIVISIBILITY BY 9



There is a well-known divisibility rule for the number **9**. In this piece, we explore that rule, and a slightly stronger version of it, and observe some consequences. Students will be able to extend the work here to a divisibility rule for **3** as well, and perhaps to a divisibility rule for **11** as well!

**EXPLODING DOTS Topic:**

**Experience 5:** Division in a  $1 \leftarrow 10$  machine.

**Suggested Grade Level:**

Middle School, High School, and All.

## DIVISIBILITY BY 9



Many people know a rule for divisibility by nine.

A number is divisible by 9 only if the sum of its digits is divisible by 9.

For example, 387261 is divisible by 9—apparently—since  $3 + 8 + 7 + 6 + 2 + 1 = 27$  is. (And if we weren't sure about the number 27, we could test that it is divisible by 9 by noting that  $2 + 7 = 9$  certainly is.)

Check:  $387261 \div 9 = 43029$  and indeed there is no remainder.

In fact, this rule can be made a little stronger.

A number leaves the same remainder upon division by 9 as does the sum of its digits.

For example, 40062 has sum of digits 12, which is 3 more than a multiple of 9, and indeed 40062 is 3 more than a multiple of 9:  $40062 = 9 \times 4059 + 3$ . Also, 77 is five more than a multiple of 9 just as  $7 + 7 = 14$ , the sum of its digits, is.

Discuss the divisibility rule for 9 with your students, and its stronger version. Have they already heard of the first rule? The second one? Have them test the rules with some examples.

Now the real question is: *Why do these rules work?*

### Some Things Students Might Notice or Ask

1. Examples do seem to suggest that the rules do hold true.
2. It is actually surprising that these rules work. If we jumble the digits of a number, the sum of digits does not change. Since 387261 is a multiple of 9, this means that 387216 and 783162 and 273613 and all other permutations of 387261 are all multiples of 9 too. That's weird, and hard to believe!

**Comment:** You and the class might choose to dwell on this observation a bit. It is certainly not generally true that you can rearrange the digits of a number and maintain its divisibility by a given factor. For example, changing the order of the digits of 512 won't always keep the number divisible by 2, and changing the order of the digits of 2864 won't always keep it a multiple of 4. So it seems awfully strange that divisibility-by-9 doesn't "care" about the order of the digits. [Have your students ever noticed it is at least true for two-digit multiples of 9? We have that 18 and 81, 27 and 72, 36 and 63, and, 45 and 54 are all multiples of 9.]

3. The second rule implies the first rule. A number is a multiple of 9 only if it leaves a remainder of zero upon division by 9. So, according to the second rule, if the sum of digits leaves a remainder of zero, that is, is a multiple of 9, then so is the original number.

Consequently, explaining the second rule automatically explains the first rule too.

## DIVISIBILITY BY 9 – AGAIN!



Here we explore a little-known technique for dividing by 9.

**EXPLODING DOTS Topic:**

**Experience 5:** Division in a  $1 \leftarrow 10$  machine.

**Suggested Grade Level:**

Middle School, High School, and All.

## DIVISIBILITY BY 9 – AGAIN!



Here's a very strange way to divide a number by 9. We'll illustrate it with a specific example.

To divide 21023 by 9, for instance, write out the partial sums of its digits, computed from left to right

$$\begin{array}{rcl}
 2 & = & 2 \\
 2 + 1 & = & 3 \\
 2 + 1 + 0 & = & 3 \\
 2 + 1 + 0 + 2 & = & 5 \\
 2 + 1 + 0 + 2 + 3 & = & 8
 \end{array}$$

and then read off the answer:

$$21023 \div 9 = 2335 \ R \ 8.$$

(And indeed,  $21023 = 9 \times 2335 + 8$ .)

In the same way,

$$\begin{array}{l}
 1221 \div 9 = 1 | 1 + 2 | 1 + 2 + 2 \ R \ 1 + 2 + 2 + 1 \\
 = 135 \ R \ 6
 \end{array}$$

and

$$\begin{array}{l}
 20000 \div 9 = 2 | 2 + 0 | 2 + 0 + 0 | 2 + 0 + 0 + 0 \ R \ 2 + 0 + 0 + 0 + 0 \\
 = 2222 \ R \ 2
 \end{array}$$

One might have to perform some explosions along the way and deal with extra-large remainders. For instance,

$$\begin{array}{l}
 5623 \div 9 = 5 | 5 + 6 | 5 + 6 + 2 \ R \ 5 + 6 + 2 + 3 \\
 = 5 | 11 | 13 \ R \ 16 \\
 = 623 \ R \ 16
 \end{array}$$

and a remainder of "16" really corresponds to one extra group of 9 and a remainder of 7. So we actually have  $5623 \div 9 = 624 \ R \ 7$ .

Discuss this algorithm with your students. Help them make sense of the examples presented here and check more examples on their own. Then ask: *Why does this curious algorithm work?*

## Some Things Students Might Notice or Ask

1. Examples do seem to suggest that this algorithm always works.
2. This is just plain weird! Really ... why does it work?
3. What if we start with a multiple of 9. Shouldn't we get a remainder of zero? But this method will never give a remainder of zero!

## DIVISIBILITY BY 7



Do you and your students know a divisibility rule for the number **7**? This piece illustrates and explains one. And after exploring it, your students will also be able to create divisibility rules for **11**, **13**, **17**, **19**, and more!

**EXPLODING DOTS Topic:**

**Experience 6:** Using dots and antidots for division in a  $1 \leftarrow 10$  machine.

**Suggested Grade Level:**

Middle School, High School, and All.

## DIVISIBILITY BY 7



Many people know divisibility rules for dividing by 2, 3, 4, 5, 8, 9, and 10, and perhaps 11 (and, hence also for 6 by using the rules for 2 and 3 together, for 12 by using the rules for 3 and 4 together, and for 15 and the like). But do you and your students know this curious divisibility rule for the number 7?

To determine whether or not a number is divisible by 7, delete its last digit and subtract double that deleted digit from what remains. The original number is divisible by 7 only if this new number is.

This procedure can be repeated until one has a number sufficiently small to be easily recognized as a multiple of 7 or not.

This is a bit hard to parse. An example helps. Let's check if 39872 is a multiple of 7.

$$\begin{array}{r} \cancel{39872} \\ - \quad 4 \\ \hline 3983 \end{array}$$

$$\begin{array}{r} \cancel{3983} \\ - \quad 6 \\ \hline 392 \end{array}$$

$$\begin{array}{r} \cancel{392} \\ - \quad 4 \\ \hline 35 \end{array}$$

Delete the final digit, the 2, and subtract double this, 4, from the 3987 that remains. This gives 3983. I can't readily tell if this is a multiple of 7, so let's do this procedure again: delete the final digit 3 and subtract double this from the remaining 398 to yield 392. It is still hard to tell whether or not we have a multiple of 7. One more run through of the procedure yields 35. This *is* a multiple of 7, and this apparently means that our original number was too!

Check:  $39872 = 5696 \times 7$ . Yes!

**By the way:** Suppose I didn't know that 35 was a multiple of 7. If I run the procedure through one more time, might I get a number I could surely recognize is divisible by 7?

Have your students practice the procedure a few times, with some beginning numbers that they know to be multiples of 7 and some known not to be.

The algorithm always seems to work. But the real question, of course, is: *Why does it work?*

### Some Things Students Might Notice or Ask

1. If you really want to know if a number is divisible by 7 or not, just get out a calculator!
2. This is just plain weird! Really ... why does it work?
3. This rule doesn't actually tell us what the answer to the division problem is. It just seems to be a YES/NO procedure.
4. If the beginning number is not a multiple of 7, does this procedure at least tell us what the remainder is upon division by 7? (Our divisibility rules for 9 did.)

## DIVISIBILITY BY 37



We've explored the divisibility rules for **3** and **9**, and **7** in other essays, and in them we also paved the way for creating divisibility rules for **11**, **13**, **17**, **19**, and more.

In this essay we adopt a different approach for discovering more divisibility rules for awkward numbers. In particular, we present in detail a divisibility rule for the number **37**.

**EXPLODING DOTS Topic:**

**Experience 6:** Using dots and antidots for division in a  $1 \leftarrow 10$  machine.

**Suggested Grade Level:**

Middle School, High School, and All.

## DIVISIBILITY BY 37



Do you and your students know this curious divisibility rule for the number 37?

To determine whether or not a number is divisible by 37, delete its first digit and add it to the digit three places further in. (Perform some carries, if necessary.) The original number is divisible by 37 only if this new number is.

This procedure can be repeated until one has a number sufficiently small to recognize as a multiple of 37 or not.

An example helps make sense of this. Let's check if 348022 is a multiple of 37.

$$\begin{array}{r}
 \cancel{3}48022 \\
 + \quad 3 \\
 \hline
 = 48322
 \end{array}
 \qquad
 \begin{array}{r}
 \cancel{4}8322 \\
 + \quad 4 \\
 \hline
 = 8362
 \end{array}
 \qquad
 \begin{array}{r}
 \cancel{8}362 \\
 + \quad 8 \\
 \hline
 = 370
 \end{array}$$

As 370 is clearly a multiple of 37, the original number 348022 must have been too!

$$\text{Check: } 348022 = 37 \times 9406.$$

As another example, this method shows that 3499 is not a multiple of 37: we recognize that  $4 \mid 9 \mid 12 = 4 \mid 10 \mid 5 = 505$  is not a multiple of 37. (How?)

This algorithm is not very practical or helpful (are you able to readily tell if a three-digit number is a multiple of 37?). But the interesting part of this is not its practical use. We want to know:

*Why does the algorithm work?*

### Some Things Students Might Notice or Ask

1. If you really want to know if a number is divisible by 37 or not, just get out a calculator! Clearly this is not a practical or efficient technique.
2. The algorithm is just weird.
3. It seems that that the algorithm doesn't actually give us the result of dividing a number by 37. It's just answering a YES/NO question.

## A STRATEGY FOR NIM



The classic game of NIM is an ancient game played with piles of pebbles. Its origin is unknown. But it was only relatively recently in 1901 that the full mathematics behind the game was explored and explained. This was done by American mathematician Charles Bouton, who also coined the name “NIM.”

A winning strategy for the game relies on writing counts of pebbles in binary and today’s puzzle is to explain why this curious winning strategy works.

**EXPLODING DOTS Topic:****Experience 2:** Understanding Binary**Suggested Grade Level:**

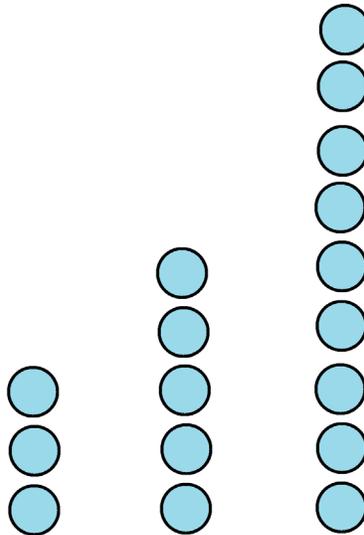
Middle School, High School, and All.

## A STRATEGY FOR NIM



Introduce the game of NIM to your students. Use counters, buttons, or some such for students to practice playing the game.

The game of NIM starts with three piles of counters on the tabletop between two seated players. One pile contains 3 counters, one 5 counters, and the third 9 counters.



Players take turns removing one or more counters from a single pile. (One is permitted to remove *all* the counters from a pile. One is required to take at least one counter.)

The person who takes the last counter—thereby leaving the tabletop empty—is the winner.

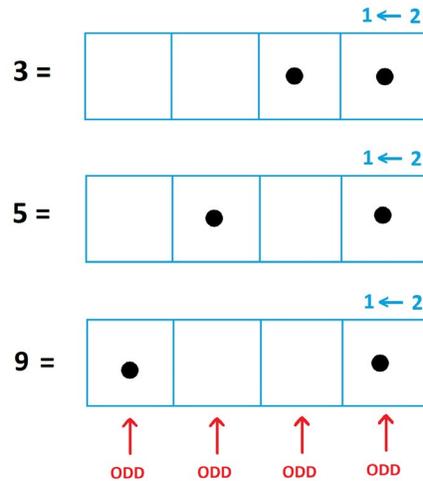
Play a few rounds of NIM with a partner just to get a feel for the game. Switch who has the first move a few times.

One can, of course, play this game with any number of piles, each containing any number of counters. To keep the conversation consistent, we'll just discuss this 3-5-9 game of NIM for now.

When the time is right, introduce material of this next section that describes a winning strategy.

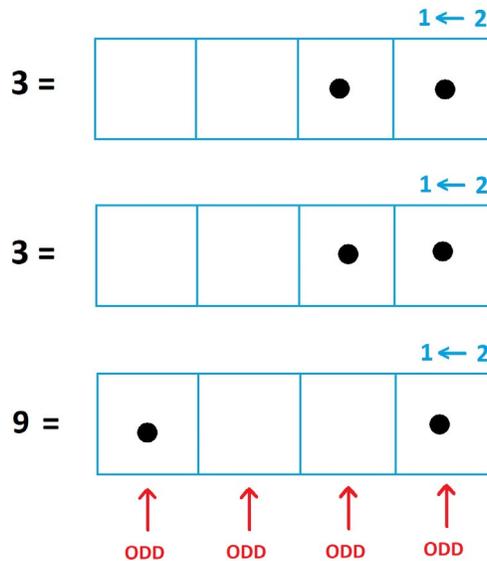
In 1901, American Mathematician Charles Bouton found a curious winning strategy for playing NIM. We present his approach here in terms of codes in a  $1 \leftarrow 2$  machine (aka binary codes).

Represent the count of pebbles in each pile in a  $1 \leftarrow 2$  machine with the machines stacked on top of one another as shown.



At present, each column in this picture contains an odd number of dots.

A NIM move will change this diagram and change the even/odd-ness of some or of all of the columns. For example, removing two pebbles from the middle pile produces this picture.

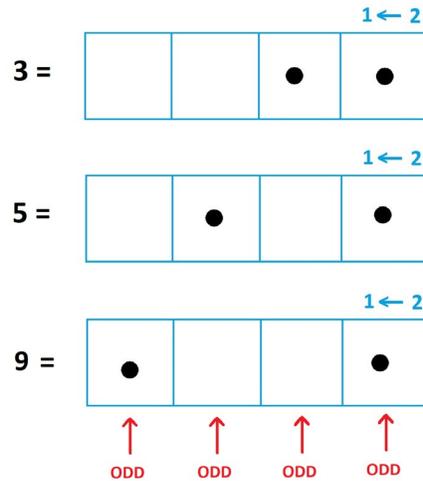


- a) Find a NIM move from the original 3-5-9 diagram that the first player can take that, instead, gives a new diagram with each column having an even count of dots.

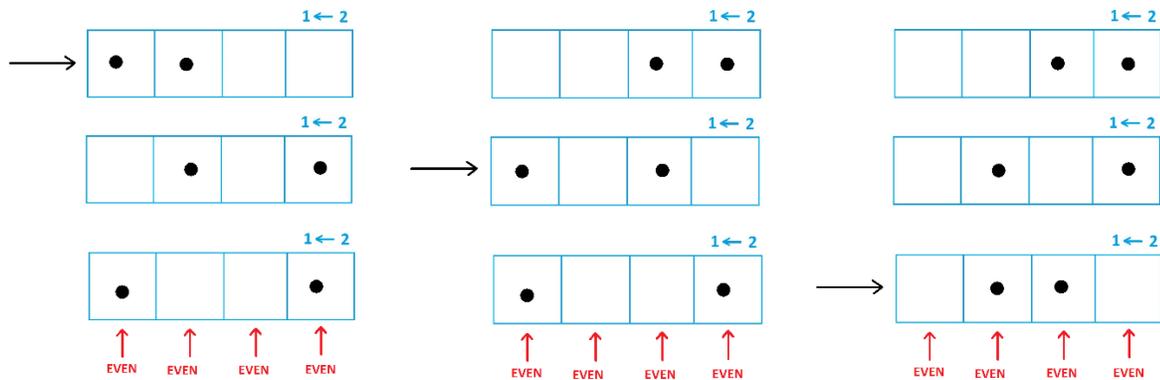
- b) From your new diagram with each column possessing an even count of dots, explain why, in whatever NIM move the second player now takes, she is sure to create a diagram with at least one column containing an odd number of dots.
- c) Prove, when presented with a diagram containing at least one column with an odd count of dots, it is always possible to make a single NIM move that gives a diagram with each column containing an even count of dots.
- d) Prove, when presented with a diagram with all columns containing an even count of dots, every NIM move is sure to create a diagram with at least one column possessing an odd count of dots.
- e) Explain why the first player of the 3-5-9 NIM game can be sure to win the game.

## Some Things Students Might Question or Notice

1. I don't know where to start. These questions seem hard!
2. So, what are we meant to do? Take this picture, change one of the rows (which means we're changing one of the piles), and get an even number of dots in each column? That doesn't seem possible!



3. Wouldn't any of these work, changing either the first, second, or third row as shown?



Oh! The first two have made piles bigger. The last one changes 3-5-9 to 3-5-6. It corresponds to taking three pebbles from the 9 pile.

So maybe it can always be done by removing pebbles from the biggest pile?

4. Part b) seems really hard: too many cases to check!

## TWO-PAN BALANCE PUZZLES



We start this classic weighing puzzle with a simple version that makes use of  $1 \leftarrow 2$  machine codes (that is, binary codes) of numbers. This, in-and-of-itself, can serve as a lovely introduction to binary.

We then move to the main puzzle itself and some extensions.

**EXPLODING DOTS Topic:**

**Experiences 2 and 9:** Using a  $1 \leftarrow 3$  machine but with the digits  $0, 1,$  and  $-1$ . (These are called “balanced ternary” codes.)

**Suggested Grade Level:**

High School and All.

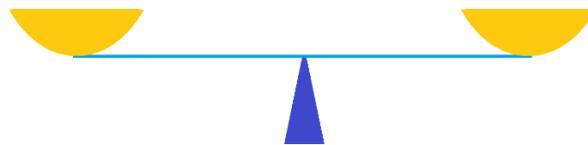
## TWO-PAN BALANCE PUZZLES



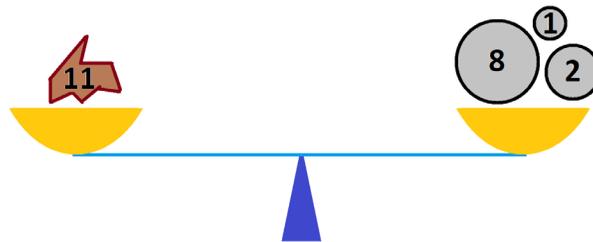
Here's a warm-up puzzle you can share with your students. (Or, for young students, this can be a lovely challenge in-and-of itself.)

Perhaps draw pictures on the board as you explain the puzzle.

You have four rocks. One weighs exactly 1 kilogram, one exactly 2 kilograms, one exactly 4 kilograms, and one exactly 8 kilograms. You also have a simple two-pan balance, with pans each big enough to hold up to four rocks.



If someone puts a rock of weight 11 kilograms, say, on the left pan, you could balance it perfectly by placing your 1 kg, 2 kg, and 8 kg rocks on the right pan.



Which rock weights in the left pan could you balance with some combination of your four rocks?

Your students will likely realize that you can balance a rock of whole number of kilograms weight from 1 kg all the way up to 15 kg with combinations of the four special rocks. From studying the codes for numbers from a  $1 \leftarrow 2$  machine, we see that every number from 1 up to 15 can be represented as a sum of some collection of the numbers 1, 2, 4, and 8.

Now add to the story ...

You decide to open a rock-weighing business. You advertise:

*Do you have a rock? And do you want to know its weight? If it weighs a whole number of kilograms between 1 kg and 15 kg, come to me! I'll figure out its weight for you.*

With trial-and-error you can figure out which combination of your four rocks balance with the customer's rock, and hence determine its weight.

Business is good for a while, and then it suddenly halts! You notice down the road that Poindexter Farklesnark has just opened a competing business. He advertises:

*Do you have a rock? And do you want to know its weight? If it weighs a whole number of kilograms between 1 kg and 40 kg, come to me! I'll figure out its weight for you.*

Customers are flocking to his business.

You see that Poindexter is also using a two-pan balance and four special rocks, but he is placing combinations of his rocks on both sides of the balance with the customer's rock. And that is how he is balancing a larger range of weight values.

Is Poindexter using four rocks of weights 1 kg, 2 kg, 4 kg, and 8 kg? Or is he using a different set of four special rocks? (If so, what are the weights of his four special rocks?)

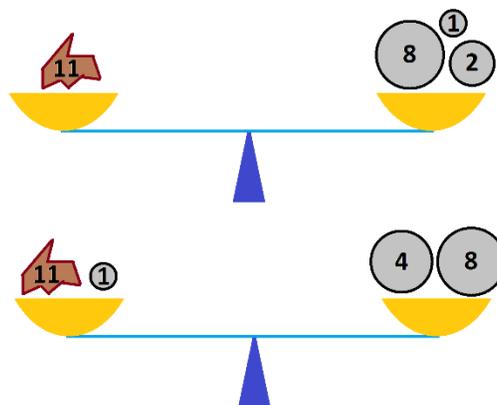
Check that Poindexter really is able to balance each of the weights 1kg up to 40 kg, inclusive, with four special rocks.

### Some Things Students Might Notice or Question

1. With rocks of weights 1 kg, 2kg, 4 kg, and 8 kg in the first business, students might wonder: Is it actually possible, "with trial-and-error," to figure out the weight of an unknown rock?

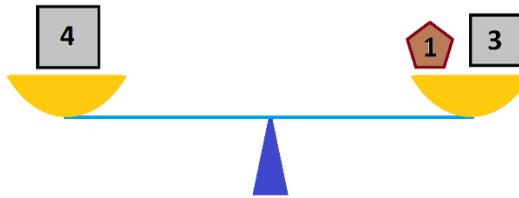
This might be worth dwelling on. Ask: How could you tell if a rock weighs more or less than 8 kg? How might you tell if a rock weighs more less than 12 kg? Could there be a general strategy to follow to determine the unknown weight of a rock?

2. With rocks of weights 1 kg, 2kg, 4 kg, and 8 kg, it is possible to balance scales in more than one way with a given customer's rock – if you allow use of both pans of the scale.



40

- Students might notice that even with use of both pans, one cannot balance a rock of weight more than 15 kg using four rocks of weights 1 kg, 2 kg, 4 kg, and 8 kg. Poindexter has four special rocks of different weights.
- Some students might argue that Poindexter must have at least one special rock of weight 1 kg: “How else can he balance a customer’s rock of 1 kg?” Other students might realize that realize that he need not! For example, with a 3 kg rock and a 4 kg rock, Poindexter can still detect a rock-weight of 1 kg.



- Some students might decide to start with the assumption that Poindexter has a 1 kg rock, just for ease, and realize that he doesn’t actually need a 2 kg rock next if he works with a 3kg rock instead. With 1 kg and 3 kg at hand, Poindexter can balance all the weights 1 kg, 2 kg, 3 kg, and 4 kg. They might wonder what the next biggest rock weight could be to get 5 kg, 6kg, 7kg, ....
- Students might say that having a rock in the same pan as the customer’s rock is like having an “anti-rock”: it subtracts from the effective weight of the customer’s rock.

## ANOTHER TWO-PAN BALANCE PUZZLE



Here's a surprising application of a  $-1 \leftarrow 2$  machine and the *negabinary* codes that result from them. These codes have a natural representation in a two-pan balance setting.

**EXPLODING DOTS Topic:**

**Experiences 2 and 9:** Having fun in bases two and negative two!

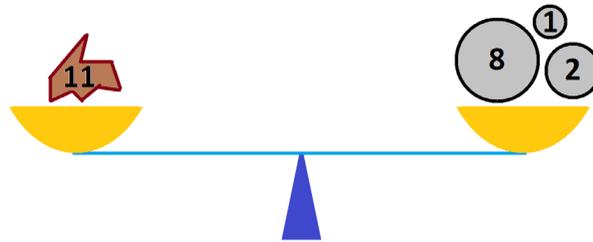
**Suggested Grade Level:**

High School, and All.

## ANOTHER TWO-PAN BALANCE PUZZLE



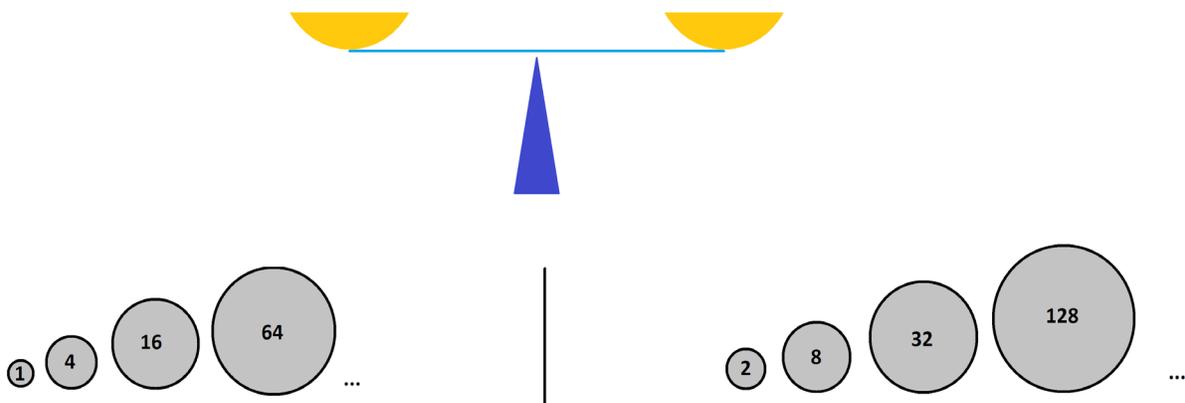
This puzzle assumes that students are familiar with the fact that every number can be written as sum of powers of two. Consequently, given a collection of rocks of weights running through the powers of two, any rock of integer weight can be placed on one side of a two-pan balance and be balance by a set of rocks from that collection



If this might not be evident to your students, present content of the “TWO-PAN SCALE PUZZLES” essay first. Then, when ready, try sharing this next challenge.

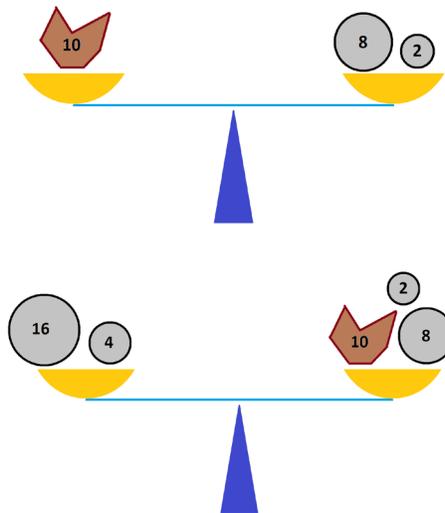
You have an infinite collection of rocks, one weighing 1 kilogram, one weighing 2 kilograms, one weighing 4 kilograms, one weighing 8 kilograms, and so on, with one rock matching each power of two in weight.

You also have a two-pan balance.



For reasons that really can't be explained you will place rocks of weights 1, 4, 16, 64, ... kilograms on the left side of the balance (and never on the right) and rocks of weights 2, 8, 32, 128, ... kilograms on the right side of the balance (and never on the left).

If someone places a rock of weight 10 kilograms on either side of the two-pan scale you realise that you can balance the system with your rocks while following these rules.



But now one wonders: Will this always be possible? Using the rocks of weights the powers of two, under the rules given, can one always create balanced system if a rock of random integer weight is placed in one of the two pans?

- a) Suppose a rock of weight 20 kilograms is placed in the right pan. Can you use the rocks of weights the powers to two to balance matters? What if the rock is placed, instead, in the left pan?
- b) Suppose a rock of weight 50 kilograms is placed in the left pan. Can you use the rocks of weights the powers to two to balance matters? What if the rock is placed, instead, in the right pan?

Let's just give the answer away.

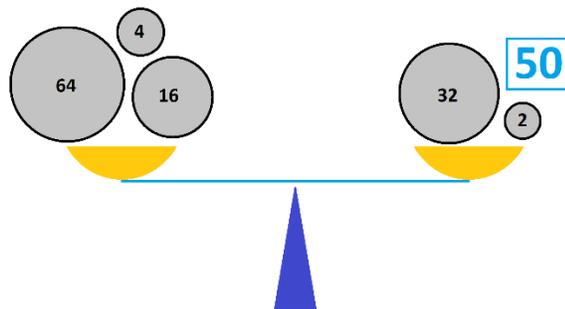
- c) Develop some convincing reasoning that if a rock of any integer weight is placed in either pan, then it is for certain possible to place rocks of weights the powers of two in the pans, while following the stated rules, to create a balanced system.

### Some Things Students Might Notice or Question

1. Via trial-and-error we see that rocks of weights 4 and 16 in the left pan balance with the rock of weight 20 in the right pan. Also, rocks of weight 8 and 32 in the right pan balance with rocks of weight 20 and 4 and 16 in the left pan.
2. Dealing with a rock of weight 50 kilograms in either pan seems annoying and hard.
3. As a sum of power of two,  $50 = 32 + 16 + 2$ . But with the 16 rock on the left and the 2 and 32 rocks on the right, this doesn't seem to mean anything?
4. You can get the equivalent of "32 on the left" by placing the 64 rock on the left and the 32 rock on the right.

Ooh! And "2 on the left" is equivalent to placing the 4 rock on the left and the 2 rock on the right.

This leads to a balanced solution with a 50-kilogram weight on the right.



5. With persistent trial-and-error, we see that

$$50 + 16 + 64 = 2 + 128.$$

This gives a balanced solution with the 50-kilogram weight on the left.

6. But this is all random trial-and-error. Is there a systematic approach?
7. The examples explored so far, 10 kilograms and 50 kilograms, are both weights an even number of kilograms. Might weights an odd number of kilograms be problematic?
8. Let's just read the solution and see if we can make sense of that!

## DIVIDING BY 101 PUZZLE



This piece is based on an American Mathematics Competition problem that asks us to determine when certain numbers are divisible by 101. It does require making use of dots and antidots within the  $1 \leftarrow 10$  machine.

Upper high-school students could see a connection to the Factor Theorem in polynomial algebra.

**EXPLODING DOTS Topic:**

**Experiences 5 and 6:** Using dots and antidots for division in a  $1 \leftarrow 10$  machine.

**Suggested Grade Level:**

High School and All.

## DIVIDING BY 101 PUZZLE



Here's a question that comes from the American Mathematics Competitions (the 2018, AMC 10B competition as question 13, in fact).

**How many of the first 2018 numbers in the sequence 101, 1001, 10001, 100001, ... are divisible by 101?**

Perhaps write this question on the board and have a class discussion with your students about it.

### Some Things Students Might Notice or Ask

1. Why the first 2018 numbers in the sequence? (If you don't mention the source of the question—that it comes from a competition that was held in the year 2018—this is indeed a strange number to pull out of the air!)
2. What is the 2018<sup>th</sup> number in the sequence?
3. Students will likely notice that the  $n$ th number in the sequence is the number created by a pair of 1s with  $n$  zeros between them.
4. The very first number in the list, at least, is divisible by 101. If we get out a calculator we can find other numbers in the list are divisible by 101 too. (Students working this way may well detect a pattern as to which ones are.)
5. Our calculators can't handle numbers with 2018 zeros between a pair of ones.

## A POURING PUZZLE



This puzzle leads on into a theory of infinitely long processes to be represented by (potentially) infinite decimals in unusual bases. The standard curriculum, of course, assume students have fully mastered thinking about decimals in early grades. But this puzzle demonstrates that there is no such thing as being “done” with a topic!

**EXPLODING DOTS Topic:**

**Experience 8:** Decimals in alternative bases

**Suggested Grade Level:**

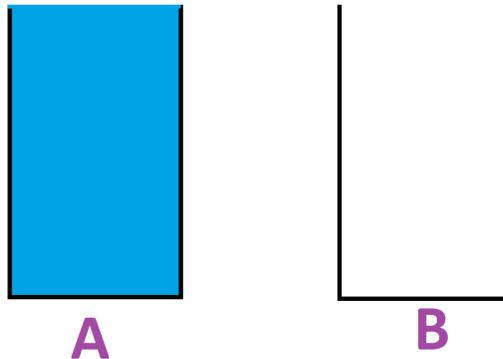
High School and All.

## A POURING PUZZLE



Here is a curious puzzle that leads into thinking about infinite processes. Share it with your students.

We have two identical containers A and B. Currently A is completely full of water and B is completely empty. We will be pouring water back and forth between the two containers.



In fact, we'll be performing very specific "pouring moves:" at any time we may either pour  $\frac{2}{3}$  of the content of container A into container B, or pour  $\frac{2}{3}$  of the content of container B into container A.

- a) After a finite sequence of pouring moves is it possible to end up with container B precisely one-quarter full?

If the answer is NO, then ...

- b) Is it possible to see an amount of water in container B that is so close to one-quarter its volume that the human eye couldn't tell the difference?

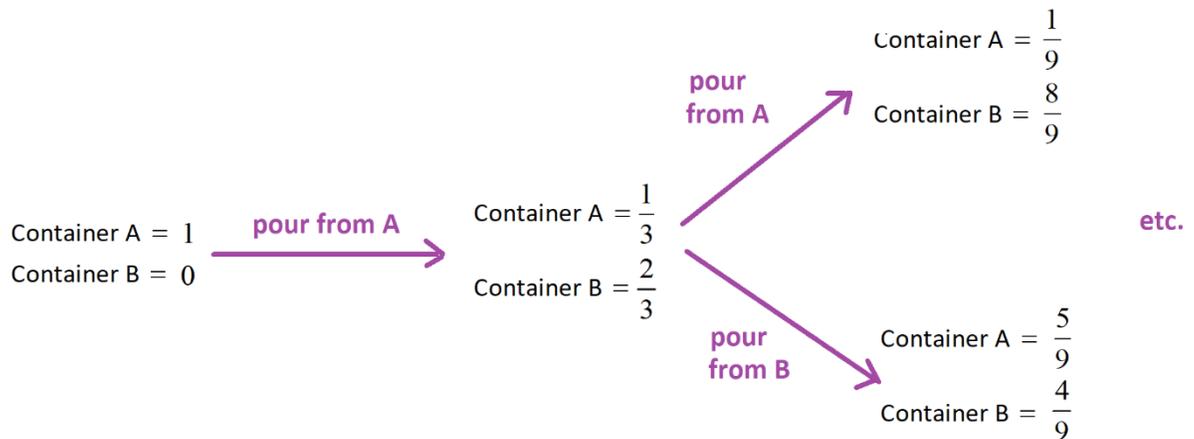
### Some Things Students Might Question or Notice

1. The fractions two-thirds and a quarter just feel “incompatible.” The answer to part a) is probably NO!
2. Given that the author of this question went on to offer part b) probably means that the answer to a) is indeed NO! (Going on pure psychology here!)
3. The author does not say one must alternate pouring directions. One could pour from the same container multiple times in a row.
4. The first move one makes must be to pour  $\frac{2}{3}$  of the content from A into B to get

$$\text{Container A} = \frac{1}{3} \text{ full}$$

$$\text{Container B} = \frac{2}{3} \text{ full.}$$

After that, there are choices.



(In the diagram, the numbers how full each container is as a fraction of its volume.)

5. Since we’re only ever taking thirds and two-thirds of fractions with denominators that are powers of three, we will never see the fraction  $\frac{1}{4}$  arise. The answer to part a) is definitely NO!

## A MANGO SHARING-AND-EATING PUZZLE



This piece is based on a USA Mathematical Talent Search question. It's a long and tricky question to read and make sense of, and this piece is chiefly about what to do when in such a challenging situation. The goal is not to solve the puzzle—though we do—but to learn how to honor one's emotional reactions and learn how to make a step of progress nonetheless. This is an important life skill!

**EXPLODING DOTS Topic:**

**Experiences 1 and 2:** Recognizing a  $1 \leftarrow 2$  machine in disguise.

**Suggested Grade Level:**

High School and All.

## A MANGO SHARING-AND-EATING PUZZLE



Consider sharing the following scary-looking problem with your students. Let them know that the point of this exercise is not to solve the puzzle—though, it will turn out we will—but to simply have an emotional reaction to it!

A group of 100 friends stands in a circle. Initially, one person has 2019 mangos, and no one else has mangos. The friends split the mangos according to the following rules:

- *sharing*: to share, a friend passes two mangos to the left and one mango to the right.
- *eating*: the mangos must also be eaten and enjoyed. However, no friend wants to be selfish and eat too many mangos. Every time a person eats a mango, they must also pass another mango to the right.

A person may only *share* if they have at least three mangos, and they may only *eat* if they have at least two mangos. The friends continue sharing and eating, until so many mangos have been eaten that no one is able to share or eat anymore.

Show that there are exactly eight people stuck with mangos, which can no longer be shared or eaten.

Source: USA Mathematical Talent Search [USAMTS](#).  
Year 31: Academic Year 2019-2020  
Round 1: Problem 4/1/31

## Some Things Students Might Say and Feel

1. This question is hard to understand: there is too much to take in!
2. The puzzle feels awfully contrived. Is it worth our while?
3. No one sits with 2019 mangos! There are never 100 people standing in a circle!
4. Ooh! I like puzzles like these.

## A Discussion to Possibly Have

Mathematics is an intensely human subject. It was invented/created for humans, by humans, to be experienced, used, and enjoyed by humans. As such, we should each be our honest human selves when doing mathematics.

There are two fundamental steps to solving any problem in math – or life, for that matter.

**Step 1:** Have an emotional reaction to the problem. Openly acknowledge your reaction.

If a problem looks hard or scary, say so! If it seems artificial and contrived, say “Why should I care?” If you don’t know what to do, acknowledge that you are flummoxed. And if you are intrigued, be intrigued. Whatever your reaction may be, consciously acknowledge your emotional response.

Then, when you are ready, take a deep breath and move to

**Step 2:** Do something. ANYTHING!

Turn the page upside down. Read the question again. Read the question backwards. Circle some words. Draw a picture. Draw a picture of a fish. Put the question aside, go for a walk, and come back later.

The point is to do something either relevant or irrelevant to the problem and work to get past an emotional impasse.

Many people just “shut down” when confronted with something confusing or scary. But by deliberately acknowledging one’s emotions and taking a step of any kind to work with them, one often finds that a first step to handling the problem falls into place, even if your first step seemed irrelevant to the task at hand.

Discuss with your students some first steps they could take with this problem.

Some directly relevant ideas they suggest could include the following.

- Read the question again.

- Read the question again and pausing after each sentence to ask: Does that sentence seem important?
- Try the question starting with 1 mango, not 2019. Then maybe 2 mangos. Then 3.
- Draw a picture of some kind demonstrating how mangos “move.”
- Start with answer 8 and somehow work backwards.
- Google the source of the problem and see if a solution was provided. (It was, but it too is very hard to parse!)

Listing such ideas is usually always fruitful. In this case, playing with small counts of mangos and drawing diagrams not only gives us a feel for what is going on, but proves to be enlightening!

**By the way:** Feel free to check out the interactive web app created by [WildThinks Blog](#) to enact this puzzle with small counts of mangos.

## THE JOSEPHUS PROBLEM



A horrific story dating back some 2000 year tells of a soldier Flavius Josephus (37 CE – ca.100 CE) being trapped in a cave with 40 fellow soldiers about to be captured by Roman forces. Rather than face the fate of being Roman slaves—or worse—they decided to commit collective suicide. They stood in a circle and counted off every third person, who was either to kill himself on the spot or, if incapable of conducting the act, be killed by his neighbors. A final soldier would be left standing who would have to commit suicide with no aid.

When the soldiers conducted this, Josephus and just one other man found themselves still standing. They decided they could not continue on and both surrendered to the Romans.

Horrors aside, this question leads to an interesting mathematical question. *Where should one stand in the circle to be the last person standing in this counting act?*

In this essay we take a less gruesome tact to this story and adjust the puzzle by counting off every second person.

**EXPLODING DOTS Topic:**

**Experience 2:** Understanding the binary codes of the  $1 \leftarrow 2$  machine.

**Suggested Grade Level:**

High School and All.

## THE JOSEPHUS PROBLEM



Present the following problem to your students. (You, of course, have the option to explain the historic origin of this problem if you wish, but it could be best to leave that aside.)

A number of students,  $N$  of them, numbered 1 through  $N$ , sit in order in a circle. Walking around the circle many times the teacher taps each student on the shoulder alternately saying the words “in” and “out” as he does each tap. Any student tapped with the word “in” stays in place and those tapped with the word “out” must leave the circle and are out of the game. Even though the circle thins out as this game is played, the teacher keeps strolling around the chairs, tapping each remaining student on the shoulder—in, out, in, out, in, out, ... —until one student remains. That lucky student wins a lifetime supply of really cool math books.

Each time the teacher plays this game with a group of students, he always starts by tapping the shoulder of student 1 with the word “in.”

As practice, consider the situation with  $N = 5$  students in a group. The teacher keeps student 1 in, sends student 2 out, keeps student 3 in, sends student 4 out, keeps student 5 in, sends student 1 out, keeps student 3 in, sends student 5 out to then leave student 3 as the winner.

Write  $W(N)$  for the number of the winning student in a game played with  $N$  students.

We have  $W(5) = 3$ .

- Check that  $W(12) = 9$  and  $W(16) = 1$ .
- If you like, complete the following table and look for patterns. Can you explain any patterns you see?

$N$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$W(N)$	1				3							9				

But here is the real question.

*Write the number  $N$  in its binary  $1 \leftarrow 2$  machine code.*

*Move the leading 1 from the front of the code to the back.*

*Then  $W(N)$  is the number with that binary code.*

For example,  $N = 12$  has code 1100. Move the front 1 to the back and get 1001, which is the code for 9. And lo and behold,  $W(12) = 9$ ! Also  $N = 5$  has code 101, move the front 1 to the back and get 011 which is the code for 3. And indeed,  $W(5) = 3$ . Whoa!

- Can you explain this mysterious connection to binary codes?

## Some Things Students Might Notice, Say, or Ask

1. No!

And this could be meant in one of two ways: that they can't explain what is going on, or that they simply reject the request to try to explain what is going on!

2. This feels like a really weird and hard puzzle.

3. "I don't get it."

The thing to do here is to conduct some more practice examples. One can write out on paper what is going on in any particular example and collect data swiftly. For example, each line here shows which student is sent out in turn for  $N = 5$  students. (In practice, one wouldn't draw a separate new line each time a student number is crossed out.)

```

1 2 3 4 5
1 2 3 4 5
1 2 3 4 5
1 2 3 4 5

```

3 wins!

It doesn't take too much effort to get some more data values.

<b>N</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>
<b>W(N)</b>	<b>1</b>	<b>1</b>	<b>3</b>	<b>1</b>	<b>3</b>	<b>5</b>	<b>7</b>	<b>1</b>	<b>3</b>	<b>5</b>

One can see that eight, for example, is 1000 in binary and moving the front one to the back gives 0001, which is the code for 1. And indeed  $W(8) = 1$ .

We can check that the claim of the puzzle keeps working!

4. Every student with an even number, 2, 4, 6, 8, 10, ..., is eliminated right away. Only odd-numbered students have a hope of winning. (Indeed, it does look like that  $W(N)$  always an odd value.)
5. In binary, the odd numbers are the ones with final digit 1.
6. "I am intrigued. But I don't know how or where to begin thinking about this!"

## AMC 10A 2020 Problem 21



Every now and then the American Mathematics Competitions present a problem ideally suited for Exploding Dots thinking. This essay is about one such problem.

**EXPLODING DOTS Topic:**

**Experiences 6 and 2:** Division of polynomials thinking in base two.

**Suggested Grade Level:**

High School and All.

## AMC 10A 2020 Problem 21



Here's a scary-looking question from the American Mathematics Competition 10A, year 2020. Perhaps copy the question onto the board or project or copy of it on a screen.

**Question 21** from AMC10A, 2020.

There exists a unique strictly increasing sequence of nonnegative integer  $a_1 < a_2 < \dots < a_k$  such that

$$\frac{2^{289} + 1}{2^{17} + 1} = 2^{a_1} + 2^{a_2} + \dots + 2^{a_k}.$$

What is  $k$ ?

Before attempting to make sense of the problem, yet alone answer, perhaps discuss with your students the emotions this problem might evoke.

### Some Things Students Might Say, Notice, or Question

4. This problem does look scary.
5. The problem is very "mathy." It seems technical and tricky.
6. It doesn't look fun. Who would want to do this?
7. It's a problem designed for tenth-graders, those who have done some geometry and algebra. Geometry probably doesn't apply, but algebra likely does. The author of this question must think that tenth-graders have the tools and means to solve it.
8. It might not be as scary as it actually looks.
9. Those numbers 17 and 289 look suspicious.
10. I don't get that " $a_1 < a_2 < \dots < a_k$ " part.
11. It's all about powers of two.

## DIVISIBILITY BY 5 in base one-and-a-half



This puzzle is really “out there.” It assumes some familiarity with the  $2 \leftarrow 3$  machine and how the codes that arise from it are representations of numbers in base one-and-a-half with the digits 0, 1, and 2. (This is Experience 9 of the Exploding Dots story.)

Very little is known about the codes of numbers from this base, including basic divisibility rules. But one can, at least, give a curious divisibility rule for 5!

**EXPLODING DOTS Topic:**

**Experience 9:** Understanding the codes of the  $2 \leftarrow 3$  machine.

**Suggested Grade Level:**

High School and All.

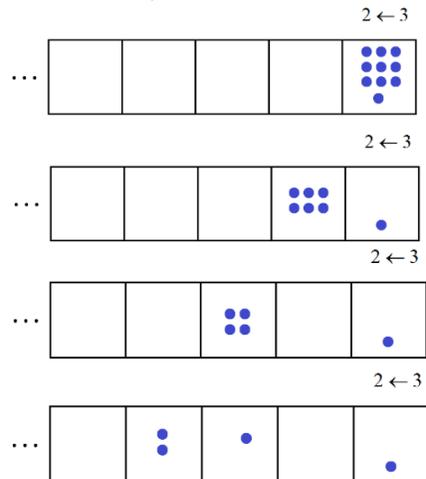
## DIVISIBILITY BY 5 in base one-and-a-half



If you and your students have played deeply with Exploding Dots and talked about the mysteries of the codes that arise in base one-and-a-half from a  $2 \leftarrow 3$  machine, perhaps try presenting this puzzle.

In a  $2 \leftarrow 3$  machine, **three** dots in any one box disappear – EXPLODE!—to be replaced with **two** dots just to their left.

For example, placing ten dots in the machine yields the code 2101 for ten.



Here are the codes for the number zero to forty from a  $2 \leftarrow 3$  machine.

0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2101120
2101	21202	212020	2101121

Prove that a number is divisible by 5 only if the alternating sum of the digits of its  $2 \leftarrow 3$  machine code is a multiple of five.

For example, twenty has code 21202 and  $2 - 1 + 2 - 0 + 2 = 5$  is a multiple of five. And forty has code 2101121 and  $2 - 1 + 0 - 1 + 1 - 2 + 1 = 0$  is a multiple of five. And eleven has code 2102 and  $2 - 1 + 0 = 2 = -1$  is not a multiple of five.

### Some Things Students Might Notice, Say, or Ask

1. What??!!!
2. This feels like the divisibility rule for eleven in ordinary base ten.
3. The  $2 \leftarrow 3$  machine code  $a | b | c | \dots | d | e$  (with each digit 0, 1, or 2) for a number  $N$  means we're writing  $N$  as

$$a\left(\frac{3}{2}\right)^k + b\left(\frac{3}{2}\right)^{k-1} + c\left(\frac{3}{2}\right)^{k-2} + \dots + d\left(\frac{3}{2}\right) + e.$$

Do we really have to mess with sums of fractions like these?

4. This is horrible!



# SOME SOLUTIONS



# CRYPTOGRAM from BRILLIANT



## SOLUTION

Let's do look at the  $1 \leftarrow 10$  machine presentation of the problem.

$$\begin{array}{|c|c|c|} \hline A & 2B & 3C \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline C & 7 & B \\ \hline \end{array}$$

$1 \leftarrow 10$

As  $A$  and  $C$  can't represent the same digit, there must at least one explosion in the tens place of the left picture that changes its leftmost box to a  $C$ . Let's do just one explosion for right now.

$$\begin{array}{|c|c|c|} \hline A+1 & 2B-10 & 3C \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline C & 7 & B \\ \hline \end{array}$$

$1 \leftarrow 10$

If we keep doing explosions from the tens place, the count of dots in that place will remain even ( $2B$ ,  $2B-10$ ,  $2B-20$ , ...). So there must be some explosions from the units place as well. Let's do just one for now.

$$\begin{array}{|c|c|c|} \hline A+1 & 2B-9 & 3C-10 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline C & 7 & B \\ \hline \end{array}$$

$1 \leftarrow 10$

Actually, pause! If we now match the places, do the equations

$$A+1 = C$$

$$2B-9 = 7$$

$$3C-10 = B$$

yield an answer? Yes. We see  $B = 8$ , then  $C = 6$ , and then  $A = 5$ , and this works!

$$\begin{array}{r} 586 \\ \phantom{5}86 \\ + \phantom{58}6 \\ \hline = 678 \end{array}$$

If we have reason to believe there is only one solution to the puzzle, then we've found it! Woohoo!  
(Do we have reason to believe this?)

## EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

**Wild Exploration 1:**

Develop some logic that explains why this is the only possible solution to the puzzle.

One approach might be to explain why there can at most one explosion from the tens place and at most one explosion from the units place in the left side of this picture.

$$\begin{array}{|c|c|c|} \hline A & 2B & 3C \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline C & 7 & B \\ \hline \end{array}$$

$1 \leftarrow 10$

**Hint:** Each of  $B$  and  $C$  is a digit from 0 to 9. The largest possible value for  $3C$  is 27, and so no more than two explosions can occur from the units place. What goes wrong with assuming it is two? The largest possible value of  $2B$  is 18. (But some dots will also appear in that tens place via the explosions from  $3C$ . Does that produce worrisome effects?)

Our work on the previous page shows that there needs to be at least one explosion of each type, so this would establish that we must indeed be dealing with exactly one explosion in each box.

$$\begin{array}{|c|c|c|} \hline A+1 & 2B-9 & 3C-10 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline C & 7 & B \\ \hline \end{array}$$

$1 \leftarrow 10$

**Wild Exploration 2:**

There are other solutions to this puzzle if we don't require  $A$ ,  $B$ , and  $C$  to be single digits. For example,  $A = 7$ ,  $B = 23$ , and  $C = 11$  works.

$$\begin{array}{r}
 7 \quad 23 \quad 11 \\
 \quad 23 \quad 11 \\
 + \quad \quad 11 \\
 \hline
 = 7 \mid 46 \mid 33 \\
 = 7 \mid 47 \mid 23 \\
 = 11 \mid 7 \mid 23
 \end{array}$$

Care to find other non-traditional solutions?

**Wild Exploration 3:** Try the remaining puzzles from <https://brilliant.org/practice/whats-the-number/>.

## MULTIPLICATION PUZZLE from BRILLIANT



## SOLUTION

Multiplying a ten-digit string of 1s with itself does gives the answer

$$1|2|3|4|5|6|7|8|9|10|9|8|7|6|5|4|3|2|1$$

if we follow the *Exploding Dots* mindset and don't worry about carrying. (Having performed smaller examples, this answer will likely be clear to many students. But performing the long multiplication to be certain, of course, does not hurt.)

With explosions, this transforms to the answer 1234567900987654321.

Multiplying a fourteen-digit string of 1s with itself gives the answer

$$1|2|3|4|5|6|7|8|9|10|11|12|13|14|13|12|11|10|9|8|7|6|5|4|3|2|1.$$

With explosions this transforms to the answer 123456790123454320987654321.

### EXTENSIONS

*Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?*

#### Wild Exploration 1:

The computation  $1 \times 1$  gives an answer that is 1 digit long.

The computation  $11 \times 11$  gives an answer that is 3 digits long.

The computation  $111 \times 111$  gives an answer that is 5 digits long.

The computation  $111111111 \times 111111111$  (with each number composed of ten 1s) gives an answer 19 digits long.

In computing  $111111111111111111 \times 111111111111111111$  (each number has twenty 1s) how many digits do you expect in the answer?

**[Warning:** Might explosions work their way all the way to the left and create an extra digit you don't first expect?]

**Wild Exploration 2:**

Do any these sets of products (or other that you might choose to play with) have interesting features?

$2 \times 2$	$2 \times 1$	$9 \times 9$
$22 \times 22$	$22 \times 11$	$99 \times 99$
$222 \times 222$	$222 \times 111$	$999 \times 999$
$2222 \times 2222$	$2222 \times 1111$	$9999 \times 9999$
$\vdots$	$\vdots$	$\vdots$

**Wild Exploration 3:**

Explore these sets of products. What do you notice?

11	101	1001
$11 \times 11$	$101 \times 101$	$1001 \times 1001$
$11 \times 11 \times 11$	$101 \times 101 \times 101$	$1001 \times 1001 \times 1001$
$11 \times 11 \times 11 \times 11$	$101 \times 101 \times 101 \times 101$	$1001 \times 1001 \times 1001 \times 1001$
$\vdots$	$\vdots$	$\vdots$

**Wild Exploration 4:** Try the remaining puzzles from <https://brilliant.org/practice/calculation-tricks/?p=2..>

## A SUM OF FRACTIONS from Brilliant



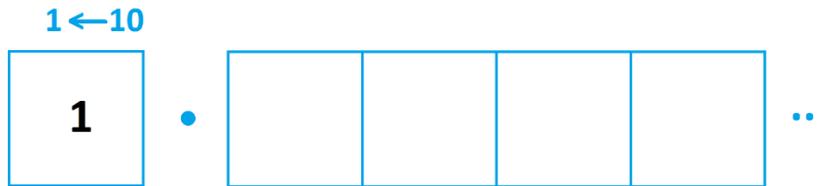
## SOLUTION

Let's determine the decimal representation of each of the fractions in the sum.

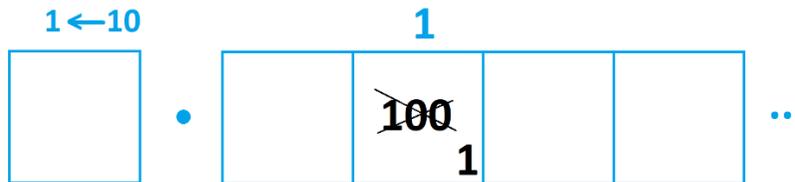
From the warm-up we know

$$\frac{1}{9} = .1111111\dots$$

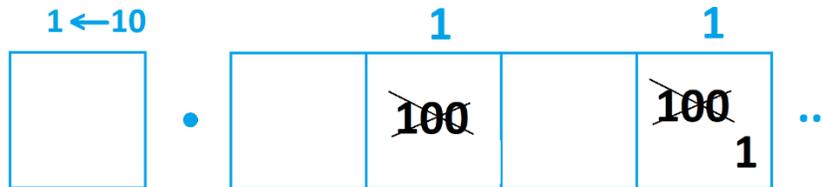
For  $\frac{1}{99}$ , we're looking for groups of 99 in this picture. (Let's write numbers rather than draw dots.)



Two unexplosions allow us to see a group of 99.



Another two unexplosions yields a next group of 99.



And so on.

We're in a cycle and we see that

$$\frac{1}{99} = .010101\dots$$

It's fun to draw the picture for  $\frac{1}{999}$  to see that sets of three unexplosions yield groups of 999 in the picture. We get

$$\frac{1}{999} = .001001001\dots$$

And in one's mind eye, the pictures for

$$\frac{1}{9999} = .000100010001\dots$$

and

$$\frac{1}{99999} = .0000100001000001\dots$$

are also clear.

Adding the five decimals gives the decimal representation for the sum  $\frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999}$ .

$$\begin{array}{r} .111111111\dots \\ .010101010\dots \\ .001001001\dots \\ .000100010\dots \\ + .000010000\dots \\ \hline = .1223231323\dots \end{array}$$

Each column will add together at least one, and at most five, 1s. Thus, each decimal digit in the sum is either 1, 2, 3, 4, or 5. We want the 24<sup>th</sup> digit.

How many 1s are in the 24<sup>th</sup> column of this sum?

There is one from the fraction  $\frac{1}{9}$ .

There is one from the fraction  $\frac{1}{99}$  as every second digit of this fraction is a 1 and 24 is a multiple of two.

There is one from the fraction  $\frac{1}{999}$  as every third digit of this fraction is a 1 and 24 is a multiple of three.

There is one from the fraction  $\frac{1}{9999}$  as every fourth digit of this fraction is a 1 and 24 is a multiple of four.

But there is not one from the fraction  $\frac{1}{99999}$  as every fifth digit of this fraction is a 1 and number 24 is not a multiple of five.

Thus, the 24<sup>th</sup> digit of  $\frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999}$  is a 4.

## EXTENSIONS

*Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?*

### Wild Exploration 1:

a) Does the digit 5 ever appear in the decimal expansion of  $\frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999}$ ? If so, where does a 5 first appear?

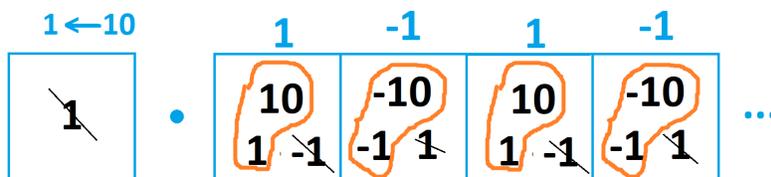
b) Here is **Puzzle 5** from <https://brilliant.org/practice/problem-solving-4/?chapter=divisibility-in-other-bases>.

$$S = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999}$$

If  $S$  is evaluated as a decimal, what is the length of its period of repetition?

**Wild Exploration 2:**

- a) Enid computed  $1 \div 11$  in a  $1 \leftarrow 10$  machine and deduced that  $\frac{1}{11} = .1|-1|1|-1|1|-1|\dots$ .



Can you make sense of what she did?

- b) Enid also deduced that  $\frac{1}{101} = .0|1|0|-1|0|1|0|-1|\dots$ . Do you see how she concluded this?
- c) When Enid was asked to find the 24<sup>th</sup> digit to the right of the decimal point in the decimal expansion of  $\frac{1}{11} + \frac{1}{101} + \frac{1}{1001} + \frac{1}{10001} + \frac{1}{100001}$  she wrote  $-4$ . How did she come up with that answer do you think?

Let's take Enid's work and rewrite her decimals the way society expects to see decimals.

- d) With unexplosions can you see that  $.1|-1|1|-1|1|-1|\dots$  equals  $.090909\dots$ ?  
(Is  $\frac{1}{11}$  indeed equal to  $.090909\dots$ ?)
- e) Convert  $\frac{1}{101} = .0|1|0|-1|0|1|0|-1|\dots$  into a decimal society expects.  
Also find the expected decimal representations of  $\frac{1}{1001}$ ,  $\frac{1}{10001}$ , and  $\frac{1}{100001}$ .
- f) **(OPTIONAL and HARD):** What is the 24<sup>th</sup> decimal digit of  $\frac{1}{11} + \frac{1}{101} + \frac{1}{1001} + \frac{1}{10001} + \frac{1}{100001}$  as an answer society expects?

**Wild Exploration 3:** Try the remaining puzzles from <https://brilliant.org/practice/problem-solving-4/?chapter=divisibility-in-other-bases>.