Exploding Dots™ Guide

Experience 11:

Grape Codes and Napier's Checkerboard

The material of this experience - Grape Codes, followed by Napier's Checkerboard - provide a perfect topic for those looking to conduct an Exploding Dots experience to a mixed audience: those who have seen some Exploding Dots before and those who have not.

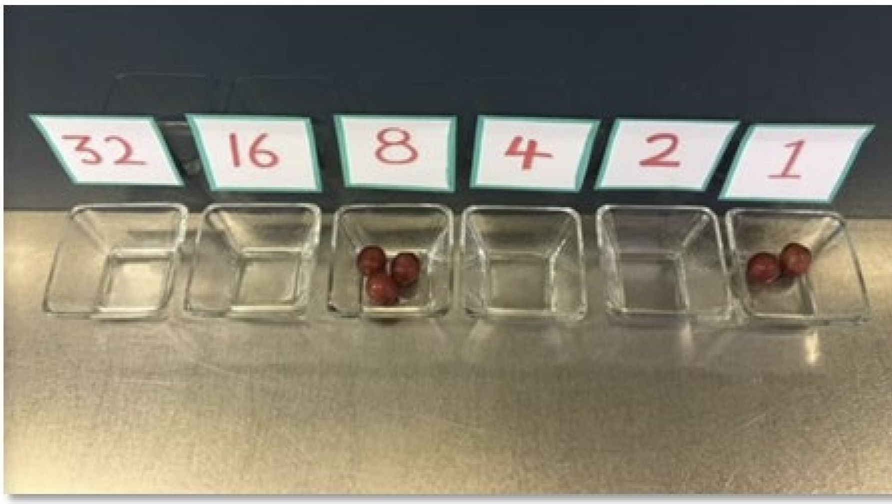
This first section is stand-alone. It can be used independently, it can be ignored and one can go straight to the story of Napier's Checkerboard, or it can be used as a segue to that material.

Year levels: All

Grape Codes

A First Video to Watch: https://youtu.be/L2hdhC6IXz8

Consider a row of dishes extending as far to the left as ever one desires, each labeled with a power of two, in order, starting from the right. In the picture I have six dishes.

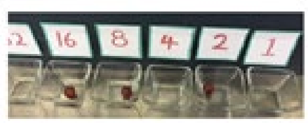


Question 1: *If I had ten dishes what would be the label of the leftmost dish?*

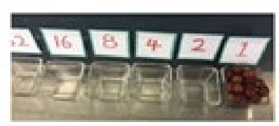
I put grapes in my dishes and when I do each grape has value given by the label of the dish in which it sits. For example, three grapes in the 8 dish and two in the 1 dish together have a total value of $8+8+8+1+1=26$. I will write $3 | 0 | 0 | 2$ as a code for twenty-six. (I'll ignore all leading zeros, that is, I won't record the empty dishes to the left of the leftmost non-empty dish.) Other "grape codes" for twenty-six are possible.

3|0|1|0

6|0|2



1|1|0|1|0



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Question 2: *There happen to be a total of* 114 *different grape codes for the number twenty-six. That is, there are* 114 *different ways to represent the number twenty-six with grapes in dishes. (This is hard to determine. A reference at the end of this section shows how I arrived at the number* 114*.) The code* $3|0|1|0$ *is one of the* 114 *ways to represent the number twenty-six. It does so with just four grapes. The code* $6|0|2$ *uses eight grapes.*

Of all 114 codes for twenty-six, is "26" (all twenty-six grapes in the 1*s bowl) the code that uses the most number of grapes? Is* $1|1|0|1|0$ *the code that uses the least number of grapes? How do you know?*

Are there two different codes for twenty-six that use the same count of grapes? Are there five different codes that use the same count of grapes?

A Second Video to Watch: https://youtu.be/XDknZLsiMxY

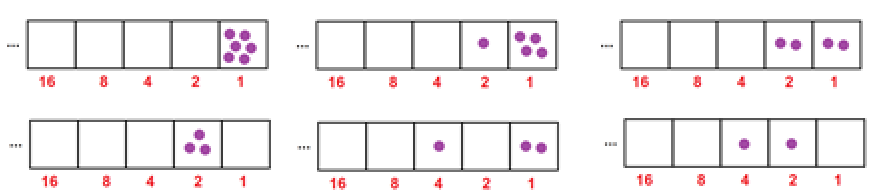
Question 3: *Here are the first few numbers that have codes using only two grapes.*

2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20, 24, 32, 33, 34, 36, …

What is the 50^{th} *number in this list?*

A Third Video to Watch: https://youtu.be/qsd58lXwLPk

Question 4: *There are* 6 *different grape codes for the number six.*



a) Show that there are also 6 *grape codes for the number seven. Actually draw diagrams for each of the codes.*

b) Is it true in general that the count of grape codes for an odd number is sure to equal the count of grape codes for the even number just before it?

THINKING OF A $1 \leftarrow 2$ MACHINE

A Fourth Video to Watch: https://youtu.be/fxo3wBek2iM

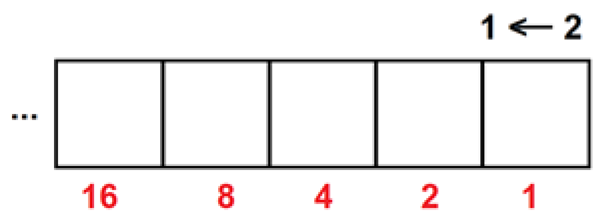
Question 5: *A code for a number with at most one grape in each dish is called a* binary code *for that number. For instance,* $1|1|0|1|0$ *is a binary code for the number twenty-six and* $1|1|0$ *is a binary code for the number six.*

a) Find a binary code for the number fifty.

b) Is every positive integer sure to have a binary code? (Read on!)

c) ***HARD CHALLENGE:*** *Could a positive integer have two different binary codes?*

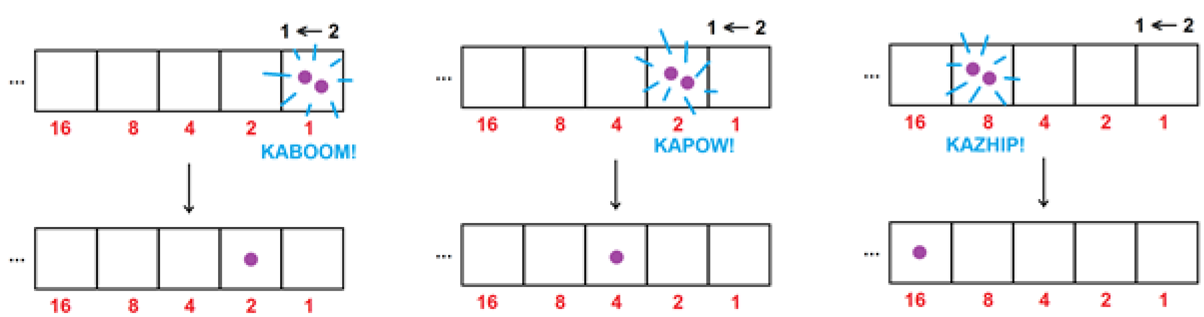
In the story of *Exploding Dots* of the Global Math Project our row of dishes is simply a $1 \leftarrow 2$ machine.



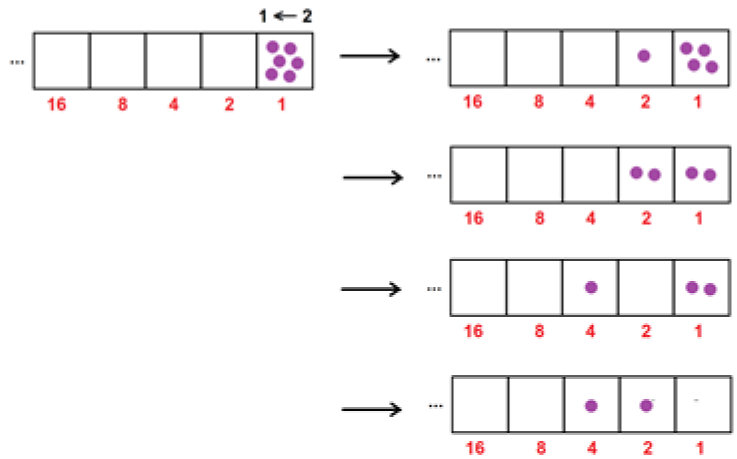
One puts in dots (or grapes) in the rightmost box and let them "explode" in the following way:

Whenever there are two dots in a box, any box, they explode and disappear - KAPOW! – to be replaced by one dot, one box to the left.

And, indeed, two dots in any one box have the same combined value as one dot just to their left.



In this way, placing a number of dots in the rightmost gives a representation of that number with at most one dot in each box. This proves that every positive integer has at least one binary code. For example, placing six dots into the machine eventually gives the binary code $1|1|0$ for the number 6.



Question 6: *Which number has binary code* $1|0|1|1$? *Which number has binary code* $1|0|1|1|0$ *and which has code* $1|0|1|1|1$?

Question 7: a) *Find the binary codes of the first twenty positive integers. What do you notice about the codes of the even numbers? The codes of the odd numbers?*

b) Anouk says she invented a divisibility rule for the number 4*:*

A number is divisible 4 precisely when its binary code ends with two zeros.

Do you agree with her rule?

c) Is there a divisibility rule for the number based 3 *on the binary code of numbers?*

Question 8: *a) Aba has a curious technique for finding the binary code of a number. She writes thenumber at the right of a page and halves it, writing the answer one place to its left, ignoring any fractions if the number was odd. She then repeats this process until she gets the number* 1*. Then she writes* 1 *under each odd number she sees and* 0 *under each even number. The result is the binary code of the original number!*

Here's her work for computing the binary code of 22*.*

1	2	5	11	22
1	0	1	1	0

Why does her technique work?

(**Hint**: Put 22 dots in a $1 \leftarrow 2$ machine and watch what happens.)

b) **FOLLOW-ON CHALLENGE:**

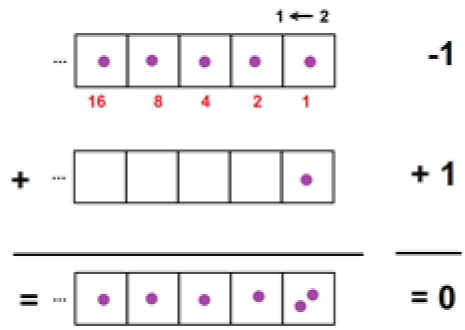
Here's a fun way to compute the product of two numbers, say, 22×13*. Write the two numbers at the head of two columns, halve the left number (ignoring in fractions) and double the right number, and repeat until the number* 1 *appears. Then cross out all the rows that have an even number on the left, and add all the numbers on the right that survive. That sum is the answer to the original product!*

Why does this method work?

~~22~~	~~x~~	~~13~~
11	x	26
5	x	52
~~2~~	~~x~~	~~104~~
1	x	208
		286

(**Hint:** $22 = 16 + 4 + 2$ and $208 + 52 + 26 = 16 \times 13 + 4 \times 13 + 2 \times 13$.)

Question 9: *Allistaire suggested that the binary code of* -1 *should be* $\cdots 1|1|1|1|1|1$ *(that is, an infinitely long string of ones going infinitely far to the left). He argued that adding one more dot to a* $1 \leftarrow 2$ *machine with a dot in each box produces, after explosions, an empty diagram: zero.*

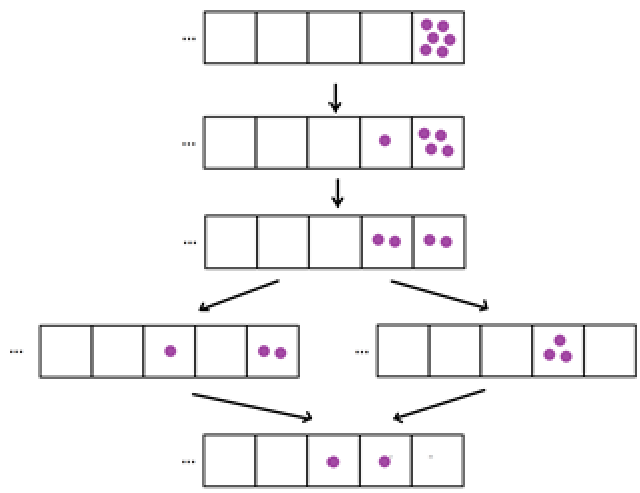


Do you agree?

Paths through Grape Codes

A Fifth Video to Watch: https://youtu.be/fcOzYK_mLWQ

The following diagram shows all the choices one can make when performing explosions on 6 dots to lead to the binary code $1|1|0$ for the number 6. The diagram also shows all ways we can represent 6 with grapes!

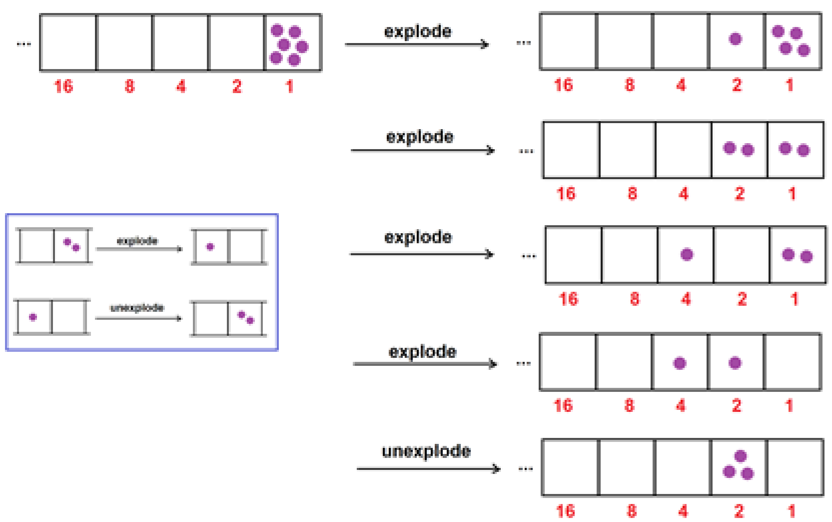


Question 10: *a) Draw an analogous diagram for* 12 *dots placed in a* $1 \leftarrow 2$ *machine. Show all the choices one can make for explosions and show that all paths lead to the same final binary code* $1|1|0|0$.

b) There are 20 *ways to represent the number* 12 *with grapes in dishes. Do all* 20 *grape codes appear in your diagram? Do all paths of explosions lead to the same binary code of* $1|1|0|0$ *?*

c) In general, when one draws a diagram of all possible explosions for N *dots placed in a* $1 \leftarrow 2$ *machine, is the diagram sure to contain all the possible codes of* N *with grapes? Do all paths lead to the same final binary code for* N *?*

Question 11: *Starting with 6 dots in a $1 \leftarrow 2$ machine, one can perform a sequence of five explosions and "unexplosions" that produces all 6 codes for 6 in terms of grapes.*



It turns out that for any positive integer N there is a sequence of explosions and unexplosions one can perform—starting with N dots in the rightmost box of a $1 \leftarrow 2$ machine—to pass through all the possible grape codes of N without repeating a code.

Starting with 12 dots in the $1 \leftarrow 2$ machine, can you find a sequence of 19 explosions and unexplosions that takes one through all 20 possible codes for 12 in terms of grapes?

Counting Grape Codes

A Sixth Video to Watch: https://youtu.be/0sCFciPUJBE

The table shows the number of different grape codes for the first few even numbers.

Number	2	4	6	8	10	12	14	16	26
# of grape codes	2	4	6	10	14	?	?	?	114

Question 12:

a) Fill in the three missing entries. Care to find a few more entries?

b) Is there a pattern to the sequence of numbers you are finding? (And can you be sure any patterns you see are genuine?)

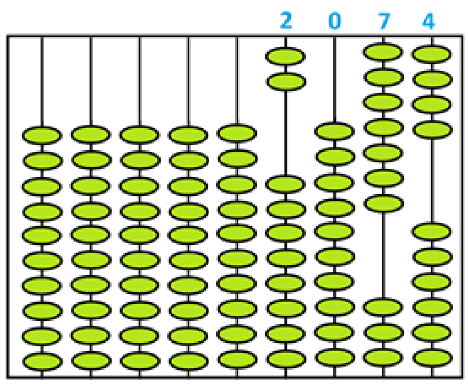
Comment: The 2018 ARML Power Question also explores these questions about codes for numbers, but not in the language of grapes nor *Exploding Dots*. To see full solutions to all the work here and its connection to the 2018 ARML Power Question, check out *Visual Graphs of Binary Representations with Exploding Dots* here:

https://gdaymath.com///wp-content/uploads/2018/07/Visual-Graphs-of-Binary-Representations-with-Exploding-Dots.pdf

NAPIER'S CHECKERBOARD

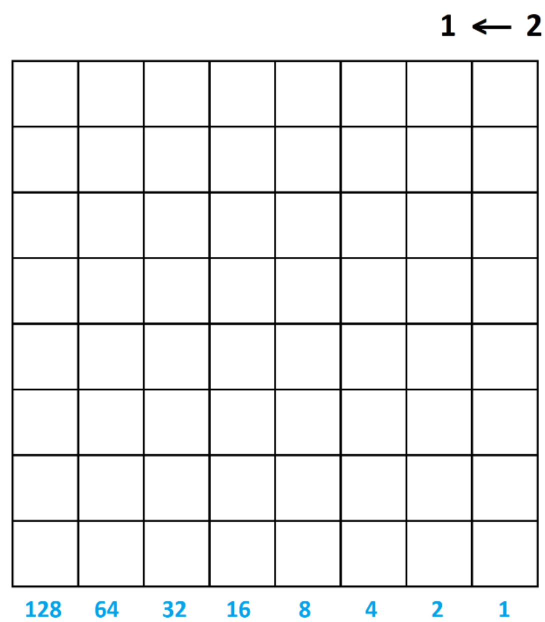
The concept of *Exploding Dots* has been around for many centuries, though not necessarily visualized as dots in boxes (and certainly not as exploding dots).

The ancient counting and arithmetic device, an *abacus*, is simply a $1 \leftarrow 10$ machine. Its simplest version is just a series of rods held in a frame with each rod holding ten beads. One slides beads up rods to represent numbers and, in performing calculations, whenever ten beads reach the top of one rod, one slides them down (they "explode") and raises one bead up on the rod one place to their left in their stead.



Comment: A more modern abacus has a cross bar with five beads on each rod below the bar and two beads above it, with each of those two beads representing a group of five. One slides beads to touch the cross bar. Thus "8," for example, is represented on a rod as three beads touching the cross bar from below and one bead touching the cross bar from above. This version of the abacus is a $1 \leftarrow 10$ machine that has a special dot (a blue dot, perhaps) that represents five dots in a box.

Five centuries ago, Scottish mathematician John Napier (1550 – 1617), best known for his invention of logarithms, actually discovered and worked with a $1 \leftarrow 2$ machine, but he found it useful to stack rows of boxes on top of one another to make a grid of squares, with each row being its own $1 \leftarrow 2$ machine.



He suggested using a physical copy this grid, a wooden board or square sheet of cloth marked into squares, and beads or counters.

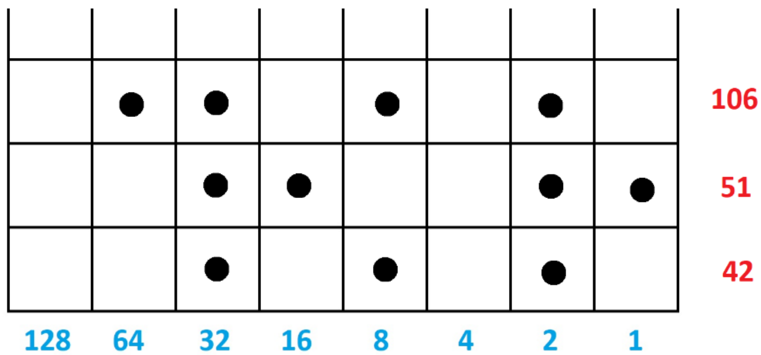
With this board, Napier showed the world how to add, subtract, multiply and divide numbers. He also felt it was useful for computing integer square roots of numbers!

Read on to see how.

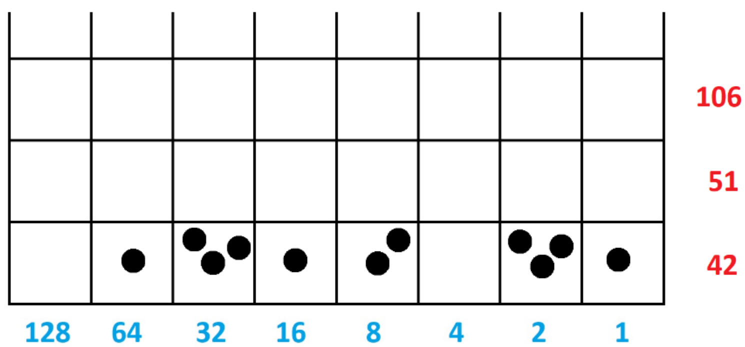
Reference: Martin Gardner wrote about this work in his article "Napier's Chessboard Abacus" which appears as chapter 8 in *Knotted Doughnuts and Other Mathematical Entertainments* (W.H.Freeman and Company, 1986) but not in terms of $1 \leftarrow 2$ machines.

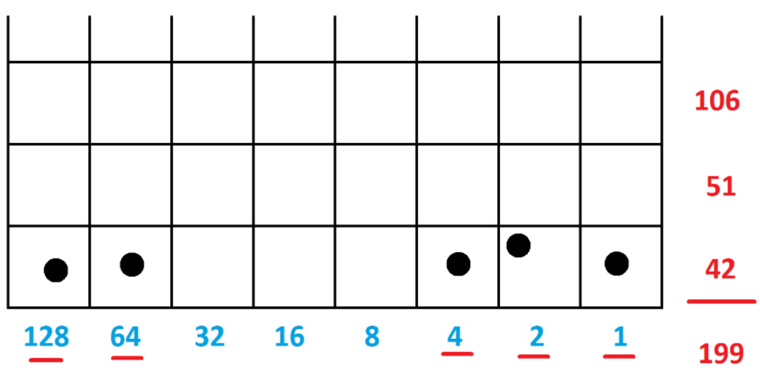
Addition

To add three numbers, say, 106, 53, and 42, represent each number on its own row of the board using counters as dots in a $1 \leftarrow 2$ machine. (Of course, Napier did not use our language of Exploding Dots and their machines, but it is clear how our language translates to actions to do with physical counters on the board.)



Then slide all the dots down to the bottom row and perform the usual $1 \leftarrow 2$ explosion rule to read off the final answer.

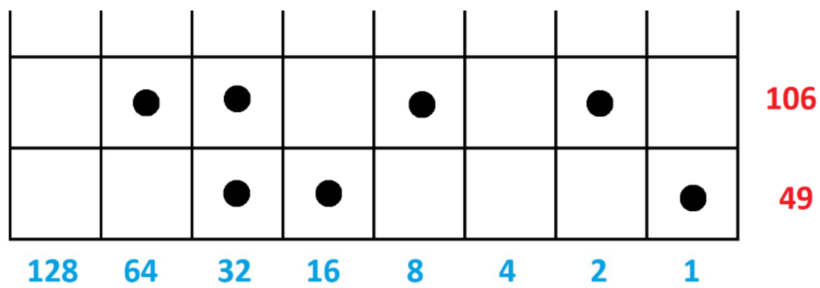




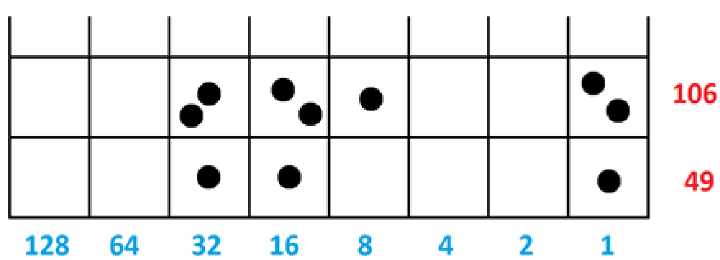
Subtraction

Napier did not introduce the notion of an antidot, but suggested performing subtraction this way instead.

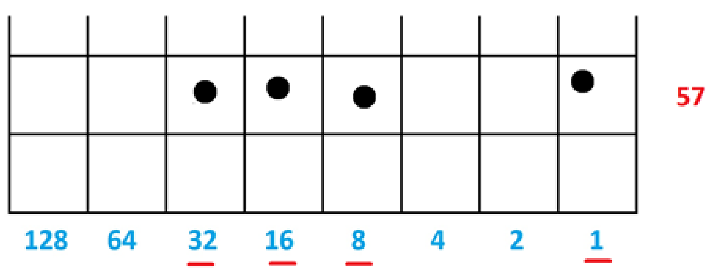
To compute $106-49$, say, represent the larger number on the second row of the board and the smaller number on the bottom row.



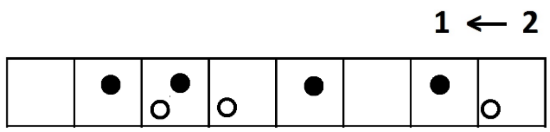
Starting at the left of the second row, perform unexplosions so that each dot in the bottom row has at least one dot above it.



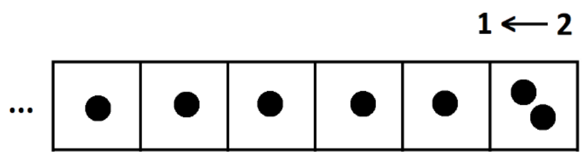
Now subtract dots from the second row, one for each dot that sits on the first row. We see the answer 57 appear.



Question: *The picture below shows how we performed subtraction in the* $1 \leftarrow 2$ *machine using antidots. Can you see a correlation of the two approaches?*

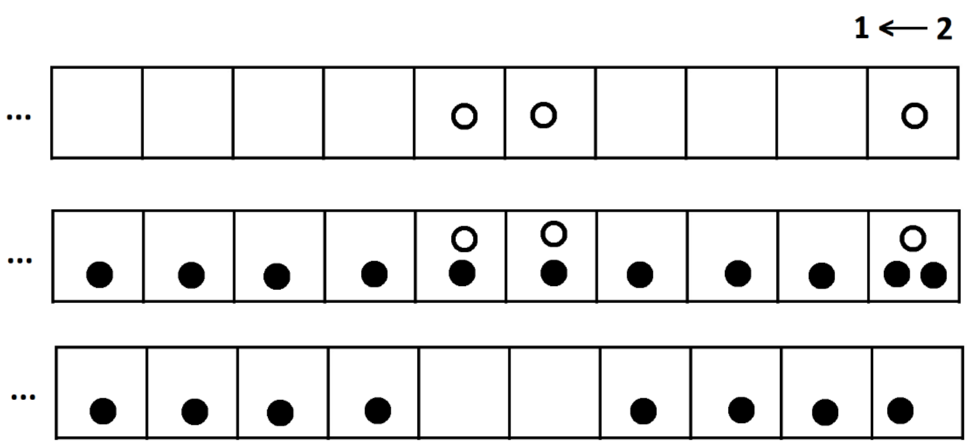


Question: *Consider a* $1 \leftarrow 2$ *machine fully loaded as shown.*



Do you see if you perform all the explosions, all the dots disappear? This shows that, in some sense, the infinitely long base-two number $\cdots 111112$ *represents the number zero. (See the chapter on* Some Unusual Mathematics for Unusual Numbers *for more on this.)*

This means we can add $\cdots 11112$ *to a picture of a negative number and not change the number. For example, we see that another representation of* -49 *in a* $1 \leftarrow 2$ *machine is* $\cdots 11111001111$.



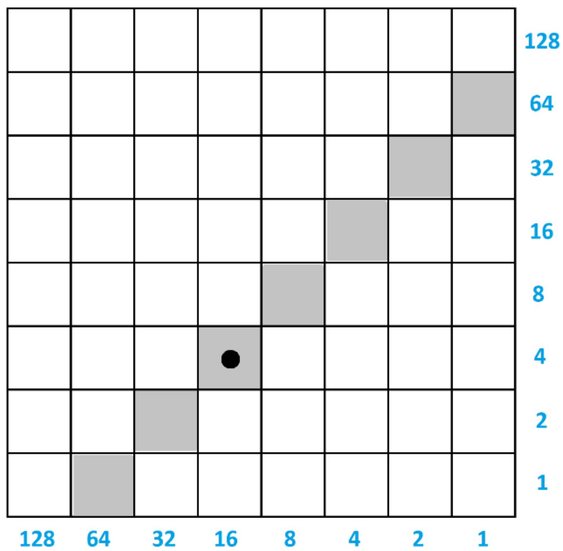
Thus every negative number can be presented in a $1 \leftarrow 2$ *machine without the use of any antidots. (The trade-off is that one must then use an infinite number of dots!)*

Compute $106 - 49$ *in Napier's checkerboard again but this time thinking of it as an addition problem,* $106 + (-49)$*, that can be presented on the board using only dots.*

Multiplication

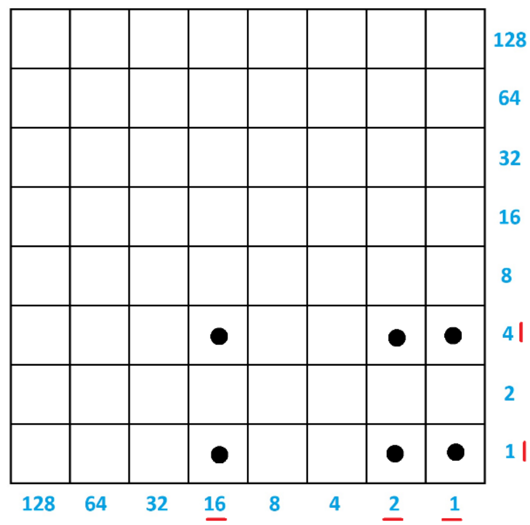
This is where Napier's brilliance starts to shine.

To perform multiplication, Napier suggested viewing the columns of the checkerboard as their own $1 \leftarrow 2$ machines! This way, each dot in a box represents a product. For example, in this picture the dot has value the product $16 \times 4 = 64$.

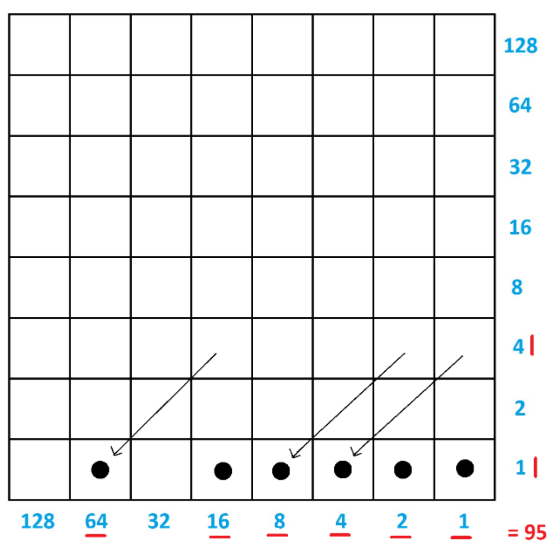


What is lovely here is that dots in the same diagonal have the same product value: $64 \times 1 = 32 \times 2 = 16 \times 4 = \cdots = 1 \times 64$. So in addition to doing $1 \leftarrow 2$ explosions horizontally and vertically, we can also slide dots diagonally and not change the total value represented by dots on the board.

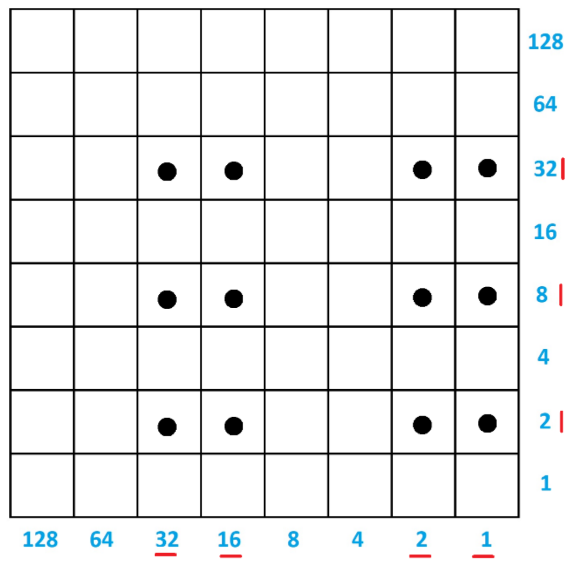
Here's a picture of one copy of 19 plus four copies of 19, that is, here is a picture of 19×5.



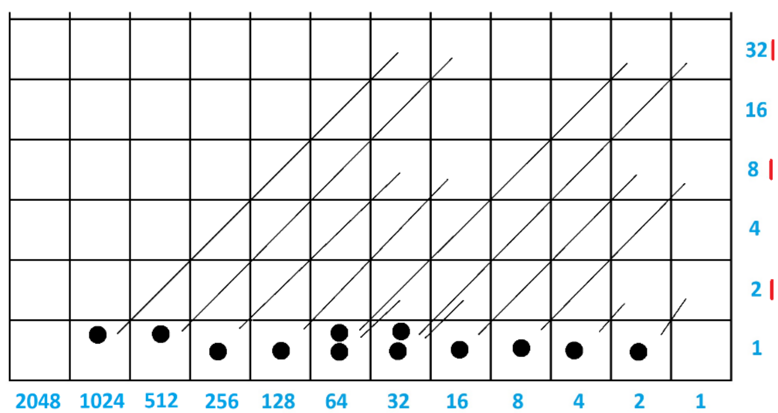
Slide each dot diagonally downward to the bottom row: this does not change the total value of the dots in the picture. The answer 95 appears.



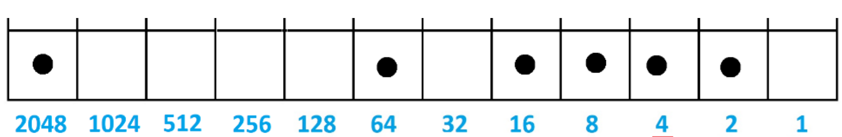
More complicated multiplication problems will likely require using a larger grid and performing some explosions. For example, here is a picture of 51×42.



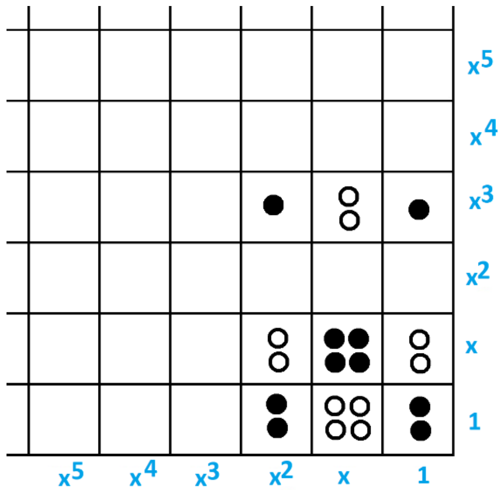
Sliding gives this picture



and the bottom row explodes to reveal the answer 2142.



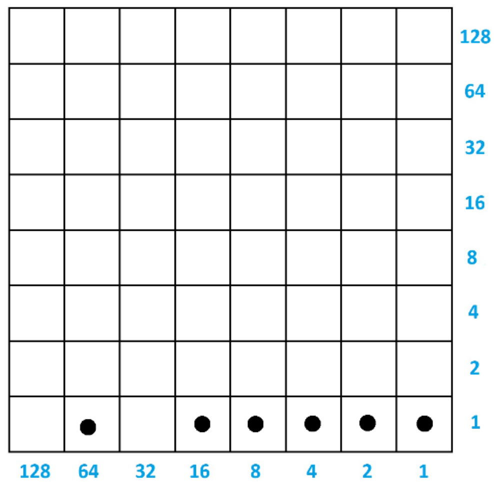
Question: *One can do polynomial multiplication with the checkerboard too! (One needs two different colored counters: one for dots and one for antidots.) Do you see how this picture represents* $(x^2-2x+1)(x^3-2x+2)$*? Do you see how to get the answer* $x^5-2x^4-x^3+6x^2-6x+2$ *from it?*



Question: *How would you display the product* $(1-x)(1+x+x^2+x^3+x^4+\cdots)$*? What answer does it give?*

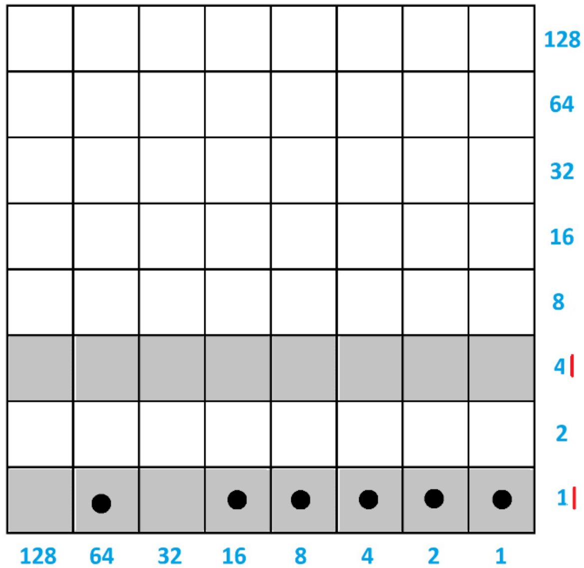
Division

Earlier we computed 19×5 and got this picture for the answer 95.

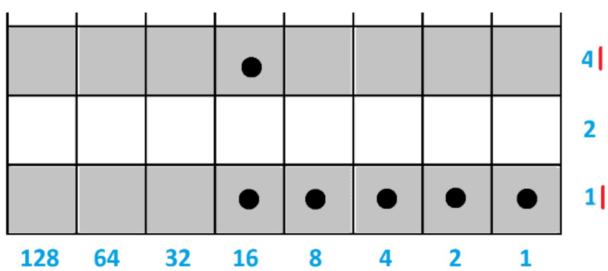


If we were given this picture of 95 first and was told that it came from a multiplication problem with one of the factors being 5, could we deduce what the other factor must have been? That is, can we use the picture to compute $95 \div 5$?

Since $5 = 4 + 1$ we will need to slide counters on this picture so that two copies of the same pattern appear in the shaded two rows.

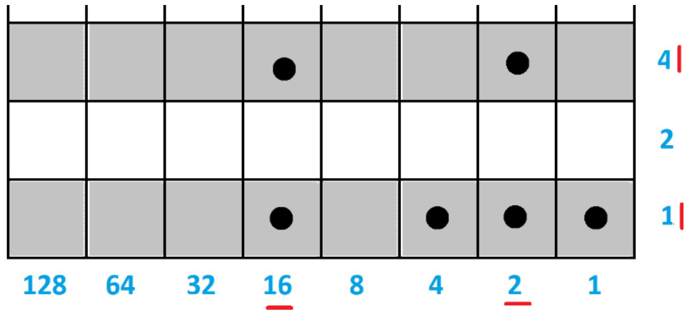


Slide the leftmost dot up to the top shaded row and we see it "completes" the 16 column. Let's not touch the counters in that column ever again.

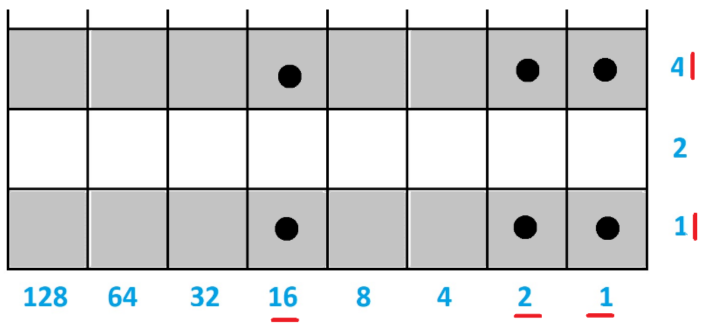


We are now left with a smaller division problem: dividing $8 + 4 + 2 + 1$ (that is, 15) by 5.

Slide its leftmost dot up to the top shaded row. This completes the 2s column and let's never touch the counters in that column again.



This leaves us with a smaller division problem to contend with: $4+1$ divided by 5. Slide its leftmost dot up to the top shaded row to complete the 1s column.



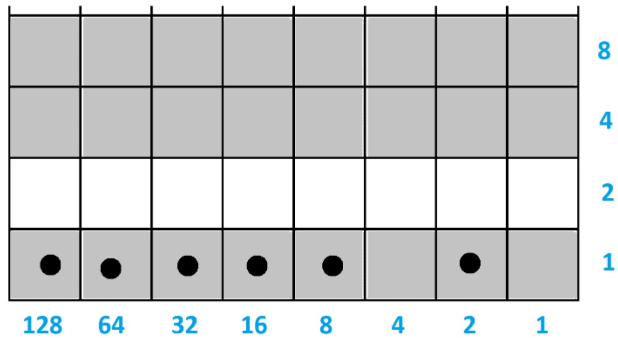
We see that we have now created the picture of 19×5 and so $95\div 5=19$.

This loosely illustrates the general principle for doing division on Napier's checkerboard:

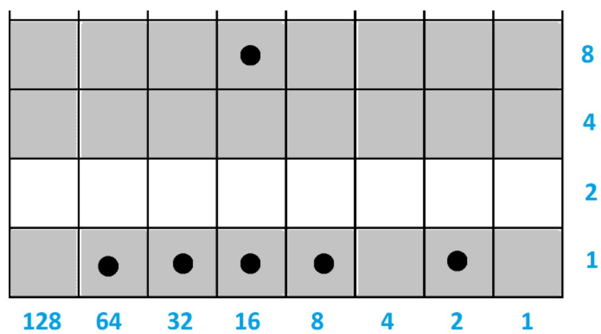
> *Represent the dividend by dots in the bottom row and the divisor by shaded rows.*
>
> *Slide the leftmost dot to the top shaded row.*
>
> *Complete the leftmost column of dots possible in some way you can (you might need to unexplode some dots) and when done never touch those dots again. What is left is a smaller division problem and repeat this procedure for the leftmost dot of that problem.*

The procedure described here is loose as our computation $95\div 5=19$ ran into no difficulties.

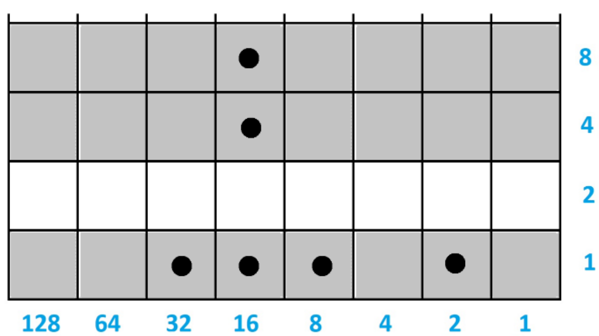
Let's try $250 \div 13$ for something more involved. Here's its setup.



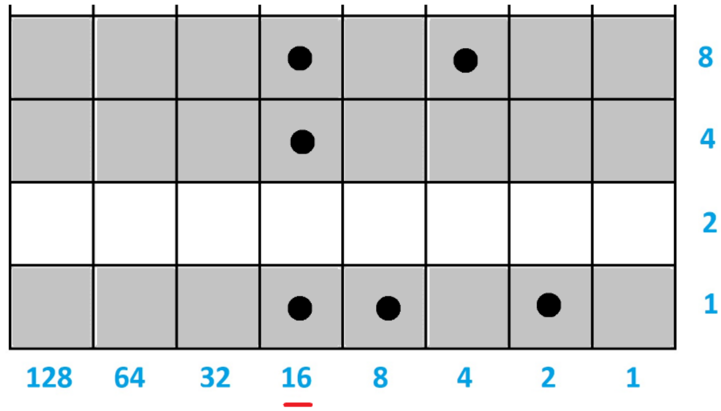
Slide the leftmost dot to the highest shaded row. Doing so shows we need to work with the 16s column, but it is not complete.



We can complete it by sliding the current leftmost dot into that column. (That's convenient!)



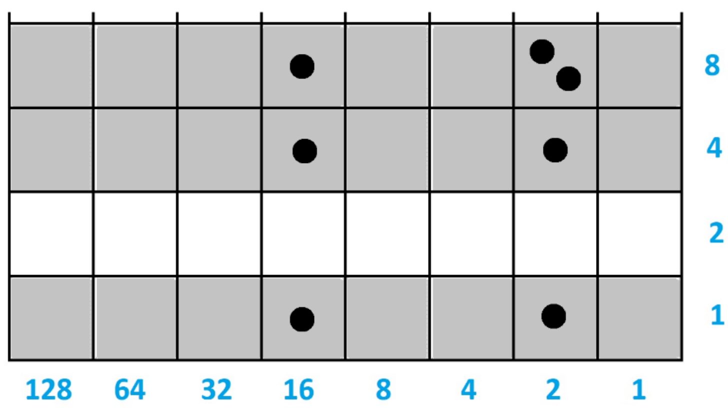
Now we have a smaller division problem to work on. Slide the leftmost dot up to the highest shaded row.



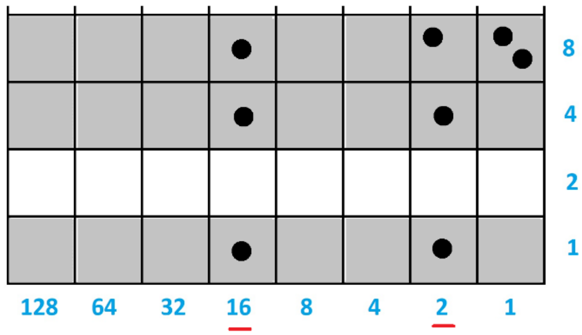
What's the leftmost column we can complete right now without ever touching those dost of the 16s column? We see that there is no means complete the 8s column. (What dot can we slide into its top?)

There is no means to complete the 4s column either. (How do we slide a dot into that 4×4 cell?)

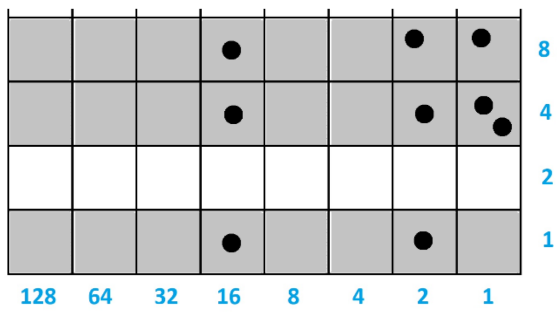
So let's work on the 2s column. I can see by sliding the dot in the 8s column and performing a (horizontal) unexplosion from the 4s column we can fill up the 2s column.

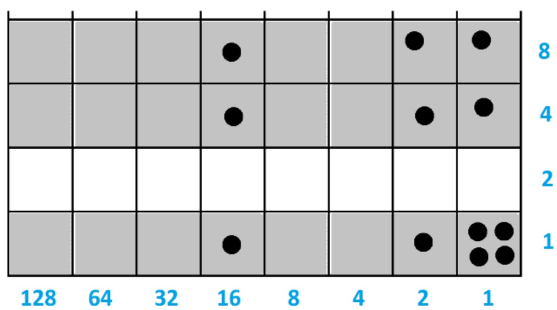


The 2s column is a bit overloaded. Let's unexplode one of the dots the top pair (horizontally).



All the action is now left in the 1s column. What can we do to make that column complete? (Remember, dots in completed columns are never to be touched again.) Let's unexplode downwards a number of times.





This does complete the 1s column, but with three ones too many.

If we had three less dots— 247 instead of 250 —then we would have, right now, a picture of 19×13 showing that $247 \div 13 = 19$. So it must be then that $250 \div 13$ has a remainder of three and so

$$250 \div 13 = 19 + \frac{3}{13}.$$

Question: *Compute* $256 \div 10$ *via Napier's method.*

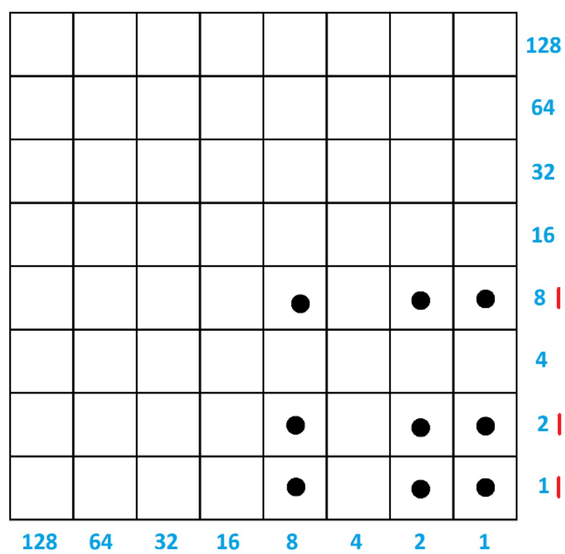
Question: *Is it possible to do polynomial division with Napier's checkerboard? (Can one compute* $\frac{1}{1-x}$ *?)*

Wild Explorations

Wild Exploration 1: Squares and Square Roots

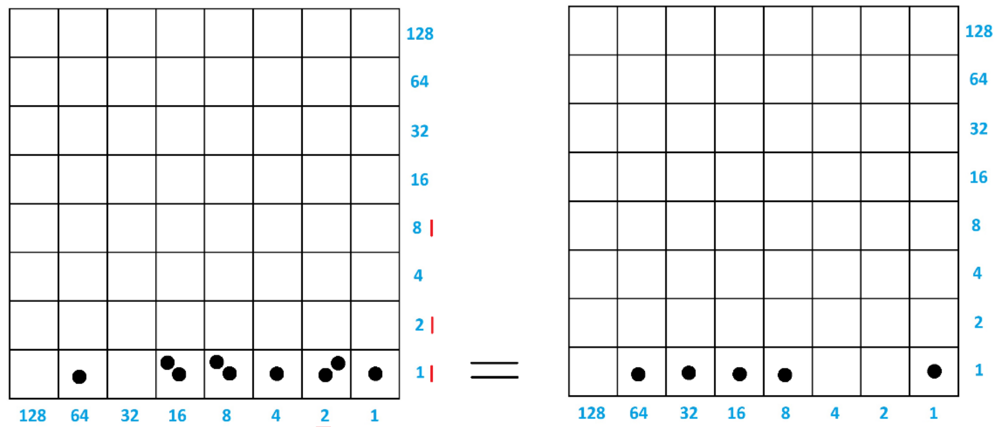
Napier claimed that his checkerboard is also capable of computing integer square root approximations to numbers. For example, his checkerboard can show that $145 = 12^2 + 1$ with 12 being the integer part of $\sqrt{150}$, and that $1000 = 31^2 + 39$ with 31 being the integer part of $\sqrt{1000}$, and so on.

To get a sense of how one might do this, consider first this picture of 11×11 to give the square number 121.



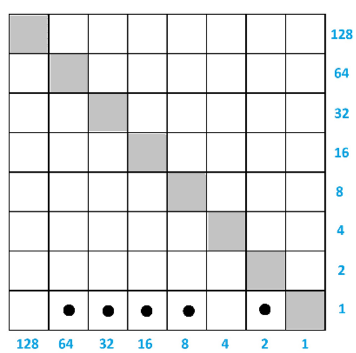
Notice the symmetry about the north-west diagonal: the picture has a pattern of dots on the bottom row, the same pattern of dots in the right column, and the same pattern appears on the diagonal too. Also, each dot in the interior of the picture sits above a dot in the bottom row and to the left of a dot in the rightmost column. All pictures of numbers squared will have such symmetry.

Sliding the dots downwards reveals 11×11 as 121.



Napier claimed that you can reverse this process and reconstruct the symmetric pattern of dots to see that 121 is eleven squared.

a) Can you indeed slide the dots that represent 121 on the bottom row diagonally upwards (or do some unexplosions and slide unexploded dots upwards) to recreate a picture of 11×11? The key is to focus on the northwest diagonal. Can you do this in a systematic way that you could explain your steps easily to a friend?



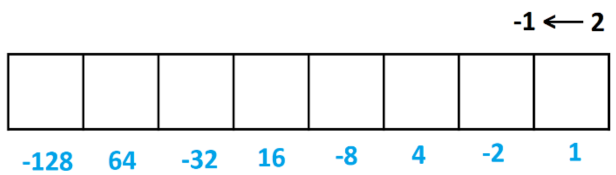
b) Use your method with the number 145 represented on the bottom row. Can you construct the picture of 12×12 (without knowing that one is looking for 12 to begin with) along with one extra dot in the 1×1 cell?

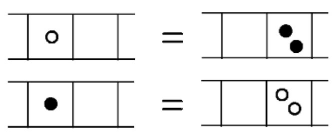
c) Use Napier's checkerboard to show that $100 = 31^2 + 39$. (Again, presume you don't know that you are looking for the number 31.)

Wild Exploration 2: Negative Numbers

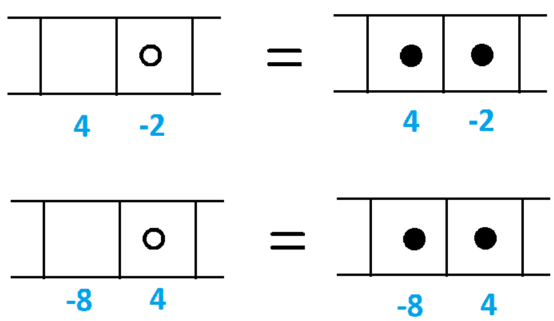
In his book *The Art of Computer Programming, Vol. 2.* (1969) Donald Knuth introduces the *negabinary system*. Here every integer, positive and negative, is represented as a sum of powers of -2 using the coefficients 0 and 1.

In the language of Exploding Dots, negabinary is a $-1 \leftarrow 2$ machine where two dots in one box explode to be replaced by one antidot, one box to the left, and similarly two antidots in a box explode to be replaced by one dot, one box to the left.



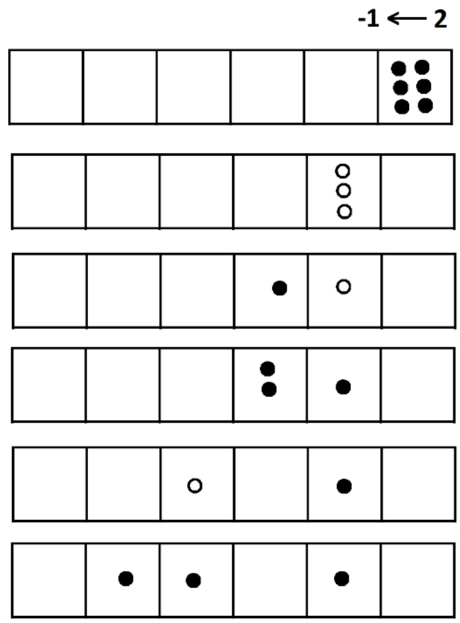


But to avoid the appearance of antidots in the representations of numbers we observe that one antidot in a box is equivalent two dots, one in the original box and one, one place to the left.

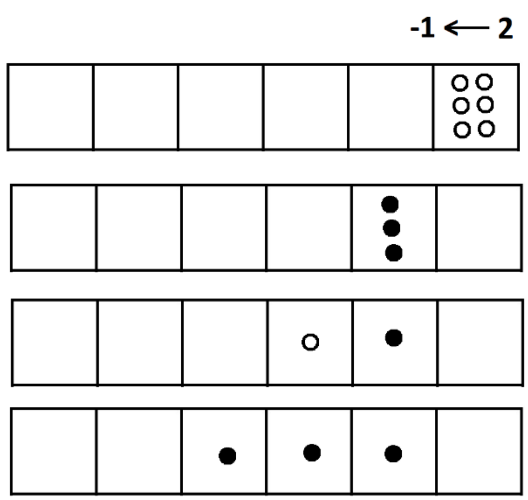


etc.

Placing six dots in the $-1 \leftarrow 2$ machine and using this convention to avoid antidots gives the negabinary code 11010 for six.



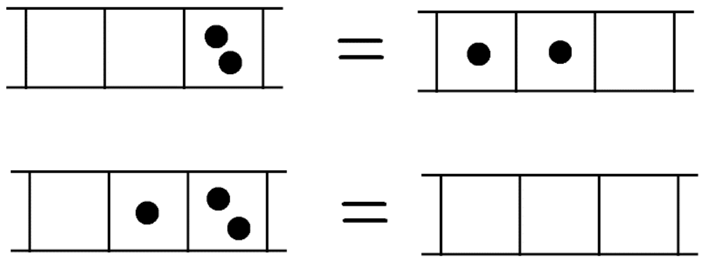
The code for -6 in this machine is 1110.



a) Work out the negabinary codes of all the integers from -10 to 10. Are there any patterns to be noticed and explained? (For example, which numbers give codes with an even number of digits? Which with an odd number of digits? Can you find a rule for divisibility by two? By three? Which numbers give palindromic codes?)

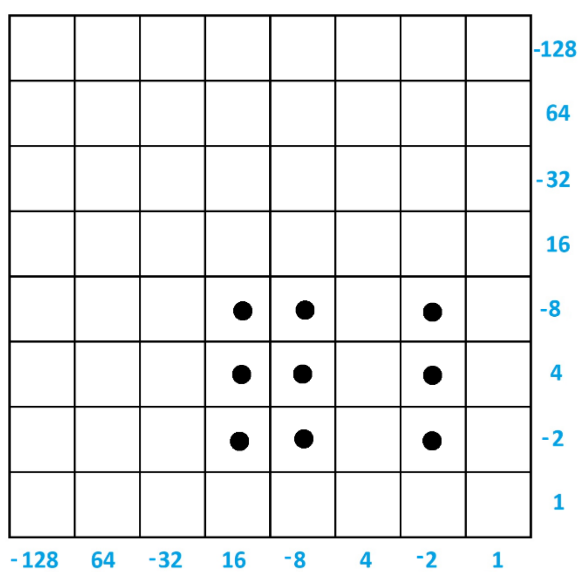
b) The $-1 \leftarrow 2$ machine shows that it is possible to represent each integer, positive or negative, in base -2 using only the digits 0 and 1 in at least one way. Prove that no integer can have two different base -2 representations using the digits 0 and 1.

Napier wasn't using two differently colored counters in his work, one for dots and one for antidots. To follow suit, note that we can rephrase the rules of the $-1 \leftarrow 2$ solely in terms of dots.



Knuth suggests using Napier's checkerboard with columns and rows labeled with values the powers of -2, representing numbers with counters in negabinary, and using the above two rules on the board (along with diagonal sliding) to manipulate pictures and thus do calculations.

For example, here is a picture of $6 \times (-6)$. Do you see how to obtain the answer -36 from it?



c) Compute $6 + (-7)$ and $6 - (-7)$ and $6 \times (-7)$ in this negabinary checkerboard.

The number negative one has code 11 in negabinary. So to change the sign of a number in negabinary we can multiply that number by 11, that is, by $10+1$. Now multiplying a number by 1 does not change the code of the number and multiplying by 10 shifts all the digits of code one place to the left. So to change the sign of a number in negabinary we can write down the code for the number, write a same code with a zero addended, and add those two codes.

d) Compute $6-(-7)$ as an addition problem of three terms: the code for 6, the code for -7, and the code for -7 with a zero addended, all added together. Did you get the same answer as you did in part c)?

e) Is there a way to perform divisions on this board too? (Try $38 \div (-13)$.)