Exploding Dots™ Guide

Experience 10:

Unusual Mathematics for Unusual Numbers

Let's now explore some unusual mathematics.
This is an advanced chapter for those feeling game!

<u>Year levels</u>: All who are game enough to go beyond the usual curriculum.

A Troubling Number for our Usual Mathematics

Here's a deep age-old question regularly asked by young math scholars across the globe.

Is $0.9999...$ *equal to one or is it not?*

This is a quantity with infinitely many 9s listed to the right of the decimal point.

Well, let's be honest, that's a dubious issue at the outset: we didn't actually list infinitely many 9s to the right. We humans never can. We might have the patience to write down twenty 9s, or two hundred 9s, or we might hire a team of two thousand people to work for twenty years to list out two billion of those 9s, but we humans will never see, actually see, an infinite number of 9s written to the right of the decimal point. We have to cheat and write an ellipsis, "$\ldots$," to say "imagine this going on forever."

So right away we should realize that the question asked is one for the mind. It is really asking:

If we were somehow God-like and could list an infinite string of 9*s to the right of the decimal point, would the quantity that results be equal to* 1*, or not?*

And one might decide that since we humans are not God-like the question is meaningless—what does "the quantity that results" mean?—and so there is no point to even giving the question consideration.

That's a valid stance to take.

But, nonetheless, this question keeps coming up. We are teased by it. And it seems we do want to try to play with it and make sense of it. Somehow it feels as though we humans, although not able to actually see and write down the infinite, can somehow envision the infinite, that we're just a small step away from having a grasp of it.

Each approximation we humans write for $0.9999...$ is close to 1, not equal to 1, in fact, just shy of 1 each time.

0.9 is less than 1.
0.99 is less than 1.
0.99999 is less than 1.
0.99999999999999 is less than 1.
$0.99999999999999999999999999999999999999999999$ is less than 1.

And some people use this to argue that $0.9999...$ too must be smaller than 1, and so not equal to 1.

But let's think a little more deeply about these numbers 0.9, 0.99, 0.999, …. They're are sequence of numbers marching rightward on the number line, heading to the number $0.9999....$, which might or might not be 1.

At the same time, we see the more 9s we list after the decimal point, the closer the numbers are to the number 1. For instance, 0.99 is one-hundredth of a unit away from 1, and 0.999999 is one-millionth of a unit away from 1.

So in fact the numbers 0.9, 0.99, 0.999, … steadily march rightward on the number line eventually entering any amount of space you might specify sitting to the left side of 1. (Want a number within a distance of $\frac{1}{337}$ from 1? 0.9999 will do!)

So could the quantity $0.9999...$ be a number sitting just shy of the number 1 and not be 1? Could there be any empty space between it and 1?

No! We just argued that the sequence of numbers 0.9, 0.99, 0.999, … will march into any space we specify to the left of 1. Thus we could conclude

There can be no space between $0.9999\ldots$ *and* 1 *on the number line, and so* $0.9999\ldots$ *must actually be* 1*.*

Well … not quite!

There is another possibility to rule out.

Could $0.9999....$ *instead be quantity sitting just to the right of* 1 *on the number line?*

This feels unlikely since all the numbers 0.9, 0.99, 0.999, … sit to the left of $1$with 1 being an upper barrier to them all.

But if we feel uneasy about a geometric argument, we can always turn to an algebraic one.

An Algebraic Argument

If you choose to believe that $0.9999....$ is a valid arithmetical quantity (that might or might not be 1), then you probably would also choose to believe that it obeys all the usual rules of arithmetic. If so, then we can conduct a lovely swift algebra argument so show that, because of these beliefs, the quantity $0.9999...$ does indeed equal 1. The argument goes as follows.

STEP 1: *Give the quantity a name.*

We'll call it F for Fredericka: $F = 0.9999\ldots.$

STEP 2: *Multiply by ten*

We obtain $10F = 9.9999\ldots.$

STEP 3: *Subtract*

Notice $10F$ and F differ only by 9 in the ones place and so $10F - F$ is 9. That is, we have

$$9F = 9$$

giving

$$F = 1.$$

The mathematics thus establishes that

$$0.9999\ldots = 1.$$

Lovely!

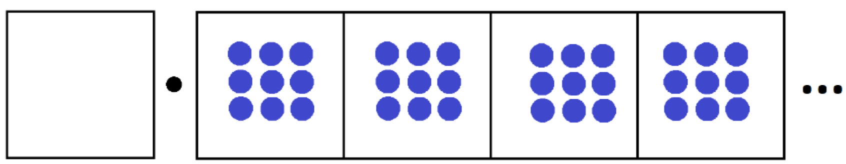
But let me be clear on what we have established here.

> ***IF** you choose to believe that $0.9999\ldots$ is a meaningful quantity in **USUAL MATHEMATICS**, then you must conclude that it equals 1.*

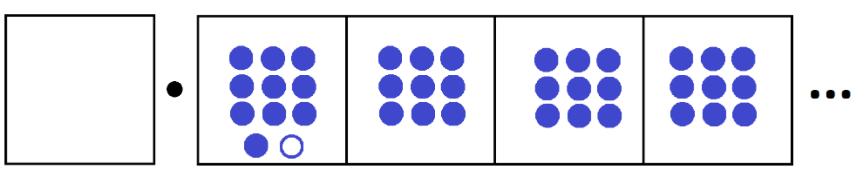
I say this because this algebraic argument can lead to philosophical woes, as we'll see in the next lesson.

Final Thought

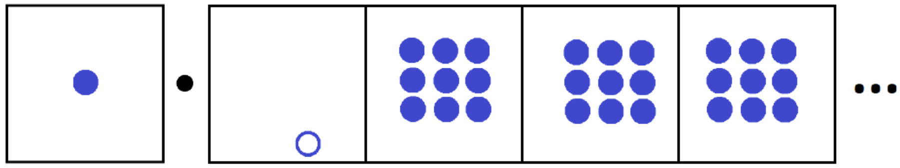
Here's a picture of $0.9999\ldots$ in a $1 \leftarrow 10$ machine with decimals.



Place a dot and an antidot in the first box. This doesn't change the number.

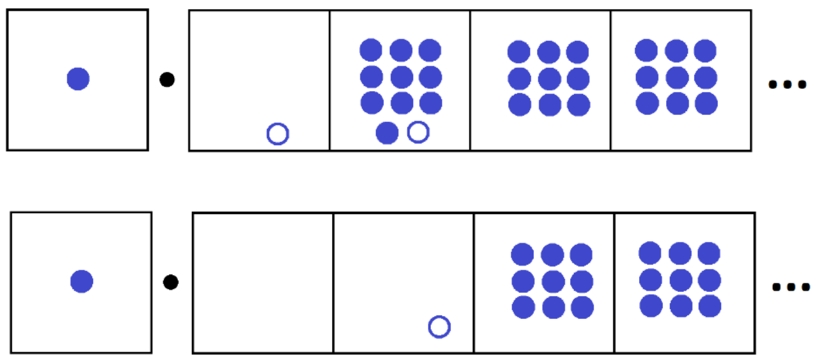


But now we can perform an explosion.



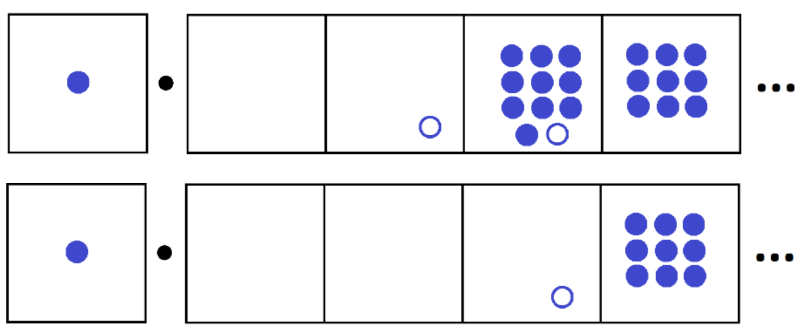
The number in this picture which looks like $1.-1\,|\,9\,|\,9\,|\,9\ldots$ is still $0.9999\ldots$.

Now add another dot-antidot pair and perform another explosion. There is now also an annihilation.



Now we have $1.0\,|\,-1\,|\,9\,|\,9\ldots$ is the same as $0.9999\ldots$.

And do this again



and again, and again, and again, forever. It looks like we are actually showing that $0.9999\ldots$ is the same as $1.0000\ldots$.

A TROUBLING NUMBER OUR USUAL MATHEMATICS REJECTS

The number $0.9999\ldots$ (if you choose to believe it is a one) has infinitely many 9s to the right of the decimal point. What if we consider the "number" with infinitely many 9s to the left of the decimal point instead?

$$\ldots 9999$$

This is a number that ends with nine. Actually, it ends with ninety-nine. Actually, it ends with nine-hundred-and-ninety-nine. And so on.

Let's apply our algebraic argument to see what value it must have.

STEP 1: ***Give the quantity a name.***

We'll call it A for Allistaire: $A = \ldots.9999$.

STEP 2: ***Multiply by ten***

We obtain $10A = \ldots.99990$.

STEP 3: ***Subtract***

We see that A and $10A$ differ by nine (it is only their final digits that differ). Looking at $A - 10A$ we get

$$-9A = 9$$

giving

$$A = -1.$$

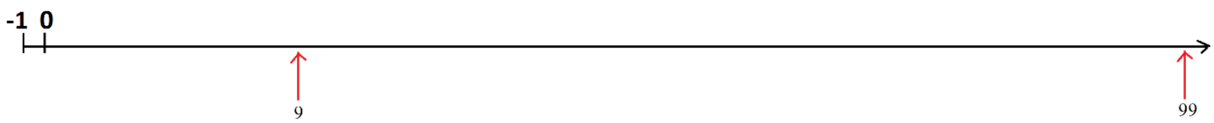
That is, our mathematics establishes that

$$\ldots 9999 = -1.$$

Apparently, if we pulled out a calculator and computed the sum $9 + 90 + 900 + 9000 + \cdots$ the calculator will show at the end of time the answer -1!

Do you believe that?

Putting it another way: On a number line, do you believe that the numbers 9, 99, 999, 9999, …. are marching closer and closer to the number -1?

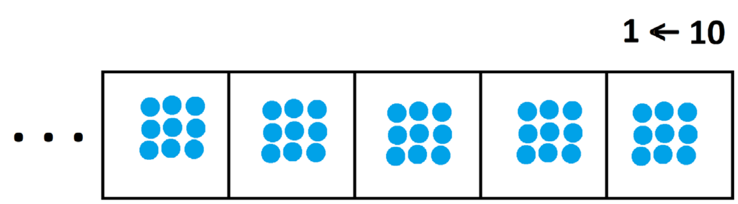


> **Challenge:** *Let's make matters worse! Consider the "number" with infinitely many 9s both to the left and to the right of the decimal point:* $...9999.9999....$. *Use the same algebraic argument to show that this equals zero. (And this makes sense, because* $...9999.9999.... = ...9999 + .9999... = -1\ + 1 = 0.$*)*

It is hard to believe that $...9999$ is a meaningful number and, moreover, it has the value -1, at least in our usual way of think about arithmetic. But remember, all we proved here is that IF we choose to say that $...9999$ is a meaningful number, then it has value -1. It is up to us to decide whether or not it is meaningful quantity in the first place. Most people say it is not and stop there and that is fine.

But this begs the question:

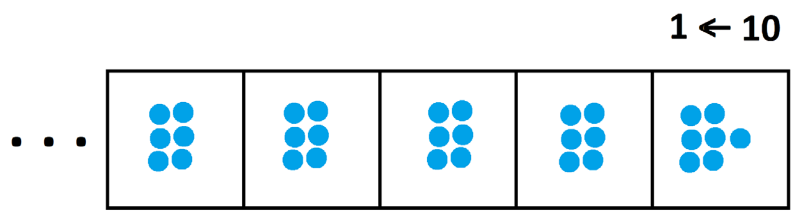
> *Is there an UNUSUAL system of arithmetic for which $...9999$ is meaningful (for which it has value -1)?*

> **Challenge:** *One might be able to argue that $...9999$ does behave like -1 in ordinary arithmetic to some degree. For example, consider performing the (very) long addition shown. Do you see the answer zero results?*
>
> $$\begin{array}{r} ...999999 \\ +\qquad\qquad 1 \\ \hline = \qquad\qquad\ \end{array}$$
>
> *If you prefer, imagine what happens if you add one dot to this loaded $1 \leftarrow 10$ machine.*
>
> 
>

Challenge: *Try this (very) long multiplication problem. Do you see that* $\ldots 666667$ *is behaving like the fraction* $\frac{1}{3}$*?*

$$\begin{array}{lr} & \ldots.66667 \\ \times & 3 \\ \hline = & \end{array}$$

If you prefer, imagine what happens if you triple the count of dots in each of these boxes.

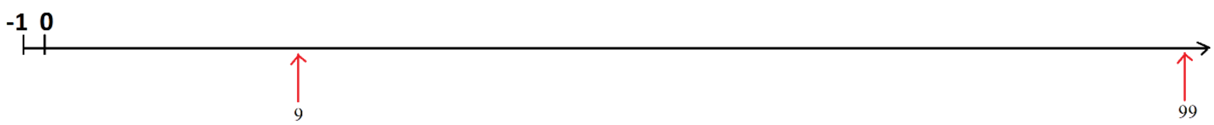


Extra: What “number” behaves like $\frac{2}{3}$?

SOME UNUSUAL MATHEMATICS FOR UNUSUAL NUMBERS

It is possible to develop an arithmetic system of numbers for which a quantity like $...9999$ actually is meaningful (and by what we proved in the previous section has value -1).

Here's one such system, one that requires we change our sense of distance between numbers on the number line. It is a system that will allow us to say, for instance, that the numbers 9, 99, 999, … are indeed marching closer and closer to -1 on the number line despite what our geometric training says!



Warping Normal Distance on the Number Line

We usually say the number 5, for instance, is a distance five from 0 on the number line because 5 is five unit lengths away from the zero. (We usually use absolute value notation for this distance: $|5| = 5$.)

And 3.7 is a distance 3.7 from 0, $|3.7| = 3.7$, because three-and-seven-tenths of a unit fit between 0 and 3.7 on the number line. And so on.

This is a very additive way of thinking about distance: adding five 1s gets you from 0 to 5; adding 3.7 1s get you from 0 to 3.7; and so on. We can say that the distance of a point a on the number line, in this thinking, is the number of 1s that go additively into a.

But much of mathematics is not only concerned with the additive properties of numbers, but also the multiplicative properties of numbers. For example, many people are interested in the prime factorizations of numbers (for example, $1000 = 2^3 5^3$ and $105 = 3 \cdot 5 \cdot 7$). There are so many unanswered questions about the prime numbers and prime factorizations still in mathematics today. These questions are, in general, very hard!

Is there are way to bring the geometry of the number line into play to possibly help with multiplicative questions? Is there a way to think about the number line itself as perhaps structured multiplicatively rather than additively?

To think about this, rather than focus on all possible factors of numbers, let's focus on one possible factor of numbers. And to keep matters relevant to our base-ten arithmetic thinking, let's focus on the number 10.

In our additive thinking for distance on the number line we use the unit of 1 and ask how many ones (additively) go into each number for its distance from 0. We now want to use the unit of 10 and ask how many times ten goes multiplicatively into each number.

What could that mean?

In the world of integers the number 0 is the most divisible number of all: it can be divided by any integer any number of times and still give an integer result (namely 0) each and every time. Focusing on our chosen factor of ten, we can divide 0 by ten once, or twice, or thirty-seven times, and still have an integer.

The number 40 is a little bit "zero-like" in this sense in that we can divide it by ten once and still have an integer. The number 1700 is more zero-like as it can be divided by ten twice and still give an integer result. A googol is very much more zero-like: it can be divided by ten one hundred times and still stay an integer.

The integer 5 is not very zero-like at all: one can't divide it by ten even once and stay an integer.

In this setting the more times ten "goes into" into a number multiplicatively, the more zero-like it is. So, in this sense, a googol is much closer to zero than 5 is.

Let's develop a distance formula that regards numbers with large powers of ten as factors as closer to zero than numbers with less counts of powers of ten as factors. There are a number of ways one might think to do this, but let's try to mimic the additive properties of the number line we are familiar with.

Normally we would say that 850 is further from zero than 85 is, and, in fact, we might even say 850 is ten times further from zero as 85 is. In our multiplicative thinking, 850 is now closer to zero than 85 is and it would be natural to have it as ten times closer.

The following formula seems a natural way to have this happen.

If a can be divided by ten a maximum of k times and remain an integer, then set $|a|_{ten} = \frac{1}{10^k}$.

For example, then, $|850|_{ten} = \frac{1}{10^1} = 0.1$ and $|8500|_{ten} = \frac{1}{10^2} = 0.01$ and $|8500000|_{ten} = \frac{1}{10^5} = 0.00001$. Also, since $|85|_{ten} = \frac{1}{10^0} = 1$ we see, indeed, that 850 is ten times closer to zero than 85 is.

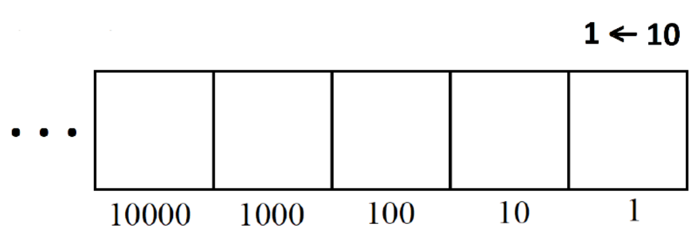
We can also measure the distance between any two numbers in this multiplicative way. For example, the distance between 3 and 33 is $|33-3|_{ten} = |30|_{ten} = \frac{1}{10^1} = 0.1$.

With this new way to measure distance, we see that

$$1, 10, 100, 1000, 10000, \ldots$$

is a sequence of numbers getting closer and closer to zero. We have $|1|_{ten} = 1$ and $|10|_{ten} = 0.1$ and $|100|_{ten} = 0.01$ and $|1000|_{ten} = 0.001$, and so on, indeed approaching a distance of zero from 0.

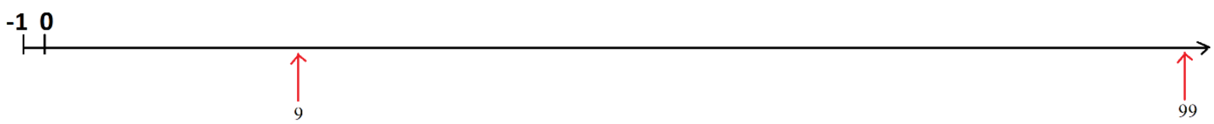
In terms of values in a $1 \leftarrow 10$ machine, we see that boxes far to the left in the machine, representing high powers of ten, are representing values very close to zero. (Before, in our additive thinking, boxes to the far right for decimals represented values very closer and closer to zero.)



Mathematicians call this way of viewing distances between the non-negative integers *ten-adic arithmetic*. (The suffix *adic* means "a counting of operations" and here we are counting factors of ten.) It is fun to think how to extend this notion of distance to fractions too, and then to all real numbers.

The number $\cdots 9999$

Let's look now at the sequence of numbers 9 and 99 and 999 and so on marching off to the right on the number line. Could they possibly be marching closer and closer to the value -1?



Yes, if by "closer" we mean this new multiplicative way to think of distance.

We have

$$
\begin{aligned}
9 &= 10 - 1 \\
99 &= 100 - 1 \\
999 &= 1000 - 1 \\
9999 &= 10000 - 1 \\
&\downarrow \\
\ldots 99999 &= 0 \quad -1 \quad = -1
\end{aligned}
$$

The numbers 9, 99, 999, 9999, … are indeed approaching the value $0-1=-1$.

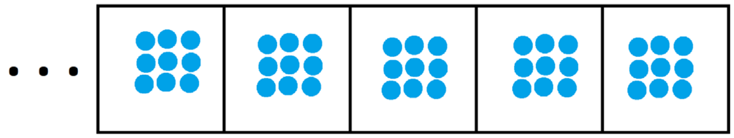
Comment: We can now justify the (very) long addition computation given below.

$$
\begin{array}{r}
\ldots 999999 \\
+ \qquad\qquad 1 \\
\hline
=\qquad\qquad\;
\end{array}
$$

We first compute $9+1=10$, and then we add 90 to this to obtain 100, and then we add 900 to obtain 1000, and so on. The further along we go with the computation the closer our results are to the number zero.

You can intuitively see this in the $1 \leftarrow 10$ machine: when you add one more dot to this loaded machine and perform the explosions, one clears away dots, pushing what remains further and further to the left where boxes have less and less significant value.

1 ← 10



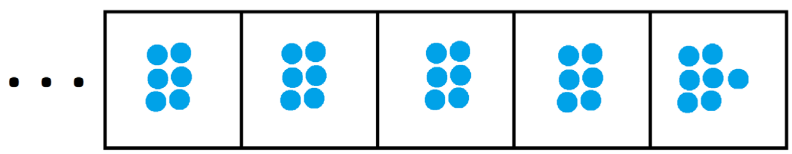
Computation of this next (very) long multiplication is justified in a similar way.

$$\begin{array}{r} \ldots.66667 \\ \times \qquad\qquad 3 \\ \hline = \qquad\qquad\quad \end{array}$$

We first compute $3 \times 7 = 21$, and then add to this 3×60 to get 201, and then add to this 3×600 to get 2001, and so on. Now the numbers 20, 200, 2000, … are getting closer and closer to zero, so the numbers $21 = 20 + 1$, $201 = 200 + 1$, $2001 = 2000 + 1$, … are getting closer and closer to 1. The further along we go with this computation, the closer our answers are to the number 1.

Again, we can intuitively see this reasoning at play by tripling all the values in this loaded $1 \leftarrow 10$ machine.

1 ← 10



And did you discover that $\cdots|3|3|3|3|4$ behaves like the fraction $\frac{2}{3}$?

Question: *Doubling* $\cdots|3|3|3|3|4$ *gives* $\cdots|6|6|6|6|8$ *which is one more than* $\cdots|6|6|6|6|7$*, which is* $\frac{1}{3}$*. Is this consistent?*

Challenge: *Show that in a* $2 \leftarrow 3$ *machine that* $\cdots 1|1|1|1|1|2$ *is negative one! Show that* $\cdots 0|1|0|1|0|1|0|2$ *when multiplied by* 5 *gives* 1*, and so represents* $\frac{1}{5}$*. (What measure of distance might we be using on the number line this time for these "numbers" to make sense?)*

Constructing Negative Integers

In our base-ten thinking with our multiplicative notion of distance on the number line, we set

$|a|_{ten} = \frac{1}{10^k}$ where k is the largest count of times a can be divided by ten and remain an integer.

And we have made sense of $\cdots 9999$ as a meaningful number with value -1.

So what's -2 in this unusual system of arithmetic?

Let's think in terms of a $1 \leftarrow 10$ machine. Since $-1 = \cdots 9|9|9|9$, and -2 is double -1, we should have

$$-2 = \cdots 18|18|18|18.$$

With explosions we get

$$\begin{aligned} -2 &= \cdots 18|18|18|18 \\ &= \cdots 18|18|19|8 \\ &= \cdots 18|19|9|8 \\ &= \cdots 19|9|9|8 \\ &= \cdots 9|9|9|8. \end{aligned}$$

And one can check that this long addition does give zero.

…99998
+ 2
=

We see now how to readily construct any negative integer. For example, we can see that adding 47 to $\cdots 9999953$ will give zero and so this latter quantity must be -47, and that adding 3000 to $\cdots 99997000$ gives zero and so this quantity must be -3000.

Challenge: *What is* -2 *in a* $2 \leftarrow 3$ *machine? What is* -5*?*

Constructing Fractions

We saw that $\cdots 66667$ is the fraction $\frac{1}{3}$: multiply this quantity by three and you get 1.

The $1 \leftarrow 10$ machine provides a natural way to compute such fractions. For example, let's find the ten-adic representation of $\frac{4}{7}$. That is, let's find a number x such that $7 \times x = 4$. Start by writing

$$x = \cdots | h | g | f | e | d | c | b | a$$

as for a $1 \leftarrow 10$ machine. Then

$$7x = \cdots | 7h | 7g | 7f | 7e | 7d | 7c | 7b | 7a.$$

We want $7a$, after explosions, to leave a 4. So we need a multiple of 7 four greater than a multiple of 10. We see that $7a = 14$ is good. So let's set $a = 2$.

$$x = \cdots | h | g | f | e | d | c | b | 2$$

$$\begin{aligned} 7x &= \cdots | 7h | 7g | 7f | 7e | 7d | 7c | 7b | 14 \\ &= \cdots | 7h | 7g | 7f | 7e | 7d | 7c | 7b+1 | 4 \end{aligned}$$

Now we want $7b+1$ to be a multiple of 10 so that all dots in that box explode to leave zero behind. This suggests $b = 7$.

$$x = \cdots | h | g | f | e | d | c | 7 | 2$$

$$\begin{aligned} 7x &= \cdots | 7h | 7g | 7f | 7e | 7d | 7c | 50 | 4 \\ &= \cdots | 7h | 7g | 7f | 7e | 7d | 7c+5 | 0 | 4 \end{aligned}$$

Now we need $7c+5$ a multiple of 10. Choose $c=5$.

$$x = \cdots | h | g | f | e | d | 5 | 7 | 2$$

$$\begin{aligned} 7x &= \cdots | 7h | 7g | 7f | 7e | 7d | 40 | 0 | 4 \\ &= \cdots | 7h | 7g | 7f | 7e | 7d+4 | 0 | 0 | 4 \end{aligned}$$

Now choose $d=8$.

$$x = \cdots | h | g | f | e | 8 | 5 | 7 | 2$$

$$\begin{aligned} 7x &= \cdots | 7h | 7g | 7f | 7e | 60 | 0 | 0 | 4 \\ &= \cdots | 7h | 7g | 7f | 7e+6 | 0 | 0 | 0 | 4 \end{aligned}$$

And then $e=2$.

$$x = \cdots | h | g | f | 2 | 8 | 5 | 7 | 2$$

$$\begin{aligned} 7x &= \cdots | 7h | 7g | 7f | 20 | 0 | 0 | 0 | 4 \\ &= \cdots | 7h | 7g | 7f+2 | 0 | 0 | 0 | 0 | 4 \end{aligned}$$

And $f=4$.

$$x = \cdots | h | g | 4 | 2 | 8 | 5 | 7 | 2$$

$$\begin{aligned} 7x &= \cdots | 7h | 7g | 30 | 0 | 0 | 0 | 0 | 4 \\ &= \cdots | 7h | 7g+3 | 0 | 0 | 0 | 0 | 0 | 4 \end{aligned}$$

And $g=1$.

$$x = \cdots | h | 1 | 4 | 2 | 8 | 5 | 7 | 2$$

$$\begin{aligned} 7x &= \cdots | 7h | 10 | 0 | 0 | 0 | 0 | 0 | 4 \\ &= \cdots | 7h+1 | 0 | 0 | 0 | 0 | 0 | 0 | 4 \end{aligned}$$

And now I am doing the same work as I did for a value b, making $7b+1$ a multiple of 10. We are in a cycle and so $x=\frac{4}{7}$ is represented as

$$\cdots 142857\ 142857\ 142857\ 2 = \overline{142857}\ 2\,.$$

Challenge: *This process felt reminiscent of the task of writing* $\frac{4}{7}$ *as a decimal in ordinary arithmetic using a* $1 \leftarrow 10$ *machine with decimals. We argued there too that the decimal representation had to fall into a cycle.*

Can you argue that the fraction $\frac{2}{13}$ *will also have a repeating ten-adic expansion?*

Challenge: *What is the ten-adic expansion of* $-\frac{4}{7}$ *?*

One approach:

Write $-\frac{4}{7}$ as $\overline{-1\,|-4\,|-2\,|-8\,|-5\,|-7}\,|-2$ and add some dots and antidot pairs to make all the terms positive.

$$\begin{aligned}&\overline{-1\,|-4\,|-2\,|-8\,|-5\,|-7}\,|-2\\ &\quad=\overline{-1\,|-4\,|-2\,|-8\,|-5\,|-7}\,|-2+(-8+8)\\ &\quad=\overline{-1\,|-4\,|-2\,|-8\,|-5\,|-7-1}\,|\,8\\ &\quad=\cdots\end{aligned}$$

A Glitch

Let's try to compute the ten-adic representation of the fraction $\frac{1}{2}$. Here we seek a number

$$x=\cdots|\,h\,|\,g\,|\,f\,|\,e\,|\,d\,|\,c\,|\,b\,|\,a$$

so that

$$2x=\cdots|\,2h\,|\,2g\,|\,2f\,|\,2e\,|\,2d\,|\,2c\,|\,2b\,|\,2a$$

equals 1.

This means we a number a so that, after explosions, $2a$ leaves a single dot. That is, we need $2a$ to be one more than a multiple of ten. This is not possible!

Challenge: *Contemplate the ten-adic expansions for* $\frac{1}{5}$ *and* $\frac{3}{10}$ *and* $\frac{2}{35}$.

In general, which fractions $\frac{p}{q}$ *seem to be problematic?*

Challenge: *Develop a general theory that if* $\frac{p}{q}$ *is a reduced fraction with* q *sharing no factor in common with ten (other than* 1*), then it is for certain possible to express* $\frac{p}{q}$ *as a ten-adic number* $\cdots hgfedcba$. *Show further that its expression is sure to fall into a repeating cycle.*

Broadening our Definition a Tad

It seems we have defined a ten-adic value to be an expression of the form $\cdots edcba$ with each digit one of the standard digits 0 through 9, allowing for non-zero digits to appear infinitely far to the left.

In this system we have the ordinary positive integers,

eg 5 is $\cdots 00005$,

the negative numbers

eg -5 is $\cdots 99995$,

and some fractions

eg $\frac{1}{3}$ is $\cdots 66667$.

But not all fractions. It turns out that the troublesome fractions are the ones $\frac{p}{q}$ which, when written in reduced form, have a denominator a multiple of 2 or 5 or both.

We can obviate this problem if we allow a ten-adic number to extend finitely far into the decimal places on the right. That is, set a ten-adic expression to be one of the form $\cdots edcba.xy\ldots z$ with each digit one of the standard digits 0 through 9, allowing for non-zero digits to appear infinitely far to the left of the decimal point, and only finitely far to its right. (After all, we do the analogous thing in ordinary arithmetic by writing $33.3333\cdots$, for example, for thirty-three and a third.)

Now we have

$$\frac{1}{2} = 0.5 \text{ is } \cdots 00000.5$$

and

$$\frac{23}{100} = 0.23 \text{ is } \cdots 000.23 .$$

We can also handle $\frac{2}{35}$ by thinking of this as

$$\frac{2}{7\times 5} = \frac{2\times 2}{7\times 10} = \frac{4/7}{10} .$$

Since $\frac{4}{7}$ is $\overline{142857}\ 2$ we must have that $\frac{2}{35}$ is $\overline{142857}.2$.

Challenge: *Show that* $\frac{1}{6} = \cdots 33333.5$ *and hence find the ten-adic expression for* $\frac{5}{12} = \frac{1}{6} + \frac{1}{4}$. *What is the ten-adic expression for* $\frac{1}{12}$ *?*

Challenge: *Explain why every fraction is now sure to have a ten-adic representation*.

Challenge: *Show that* $\overline{53}23 = \ldots 5353535323$ *is the number* $-\frac{1000}{33}$ *in ten-adic arithmetic.* (**Hint**: Multiply the quantity by 100 and subtract.)

One can use the technique of this question to show that every ten-adic number that eventually falls into a cycle going leftwards is a rational number.

Challenge: *In ordinary arithmetic, the quantity* $0.abcabcabc\cdots = 0.\overline{abc}$ *is the fraction* $\frac{abc}{999}$. *We see this by setting* $x = 0.abcabcabc\cdots$ *and noticing that* $1000x = abc.abcabcabc\cdots$. *Subtracting then yields* $999x = abc$.

Show that the same algebra applied to the ten-adic number $\cdots abc\,abc\,abc = \overline{abc}$ *shows that it this number has value* $-\frac{abc}{999}$.

In fact, prove the following general result. Suppose b_1, b_2, …, b_k *are single digits.*

If $0.\overline{b_1b_2\ldots b_k}$ *is the fraction* $\frac{p}{q}$ *in ordinary arithmetic,*

then $\overline{b_1b_2\ldots b_k}$ *is the fraction* $-\frac{p}{q}$ *in ten-adic arithmetic, and vice versa.*

Challenge: *Explore a theory of "3/2-adic" representations of fractions using a* $2 \leftarrow 3$ *machine.*

A SERIOUS FLAW OF TEN-ADIC NUMBERS

With the ten-adic numbers we can represent all integers and fractions. But did you notice that all the numbers we presented so far have repeating cycles?

We can also consider numbers in this system, with infinitely many digits to the left, that don't fall into repeating cycles. These must correspond to irrational numbers. (And possibly other new types of numbers?)

We can add and multiply ten-adic numbers. For example, we have seen $\ldots 6667$ is $\frac{1}{3}$ and $\cdots 9999$ is -1. We can compute their sum to see an answer indeed one less than $\ldots 6667$. (And $\ldots 6666$ is $-\frac{6}{9} = -\frac{2}{3} = \frac{1}{3} - 1$.)

$$\begin{array}{rccccc} & \ldots & 6 & 6 & 6 & 7 \\ + & \ldots & 9 & 9 & 9 & 9 \\ \hline = & \ldots & 15 | & 15 | & 15 | & 16 = \ldots 6666 \end{array}$$

And we can compute their product to see the answer $\ldots 3333$, which is indeed $-\frac{1}{3}$.

$$\begin{array}{rccccc} & \ldots & 6 & 6 & 6 & 7 \\ \times & \ldots & 9 & 9 & 9 & 9 \\ \hline = & \ldots & 54\ | & 54\ | & 54\ | & 63 \\ & \ldots & 54\ | & 54\ | & 63\ | & 0 \\ & \ldots & 54\ | & 63\ | & 0\ | & 0 \\ & \ldots & 63\ | & 0\ | & 0\ | & 0 \\ & & \vdots & & & \\ \hline = & \ldots & 225 | & 171 | & 117 | & 63 = \ldots 3|3|3|3 \end{array}$$

And since we know how to make negative ten-adic numbers and fractions as ten-adic numbers, we can also subtract (add the negative) and divide (multiply by a fraction) ten-adic numbers.

Well, almost. We can't divide by some ten-adic numbers. There's a flaw in the ten-adic system.

Here's an example of the problem. We have that

$2\times 5=10$ is a number close to zero.
$2^2\times 5^2=100$ is a number closer to zero.
$2^3\times 5^3=1000$ is a number even closer to zero.
And so on.

It is possible to construct a non-zero ten-adic number N that behaves like an infinitely large power of two and a non-zero ten-adic number M that behaves like an infinitely large power of five, so their product then, $N\times M$, is so close to zero that it actually is zero!

$$N\times M=0$$

Non-zero numbers that multiply to zero don't exist ordinary arithmetic, but they do in the ten-adic system. This means we can't divide by some non-zero numbers, like N and M in this arithmetic. (If dividing by N is possible, then divide the equation $N\times M=0$ through by N and get the contradictory statement that $M=0$.)

How might one construct these numbers N and M? It's a bit tricky, but here's the gist of it.

Here are the first few powers of five:

5, 25, 125, 625, 3125, 15625, 78125, …

All of these powers end in 5.

Infinitely of them actually end in 25.

Actually, one can verify that infinitely of then actually end in 125. (Multiply any one that does by 25 to get another one.)

Actually, infinitely many of them end in 3125. (Multiply any on that does by 625 to find another one that does.)

And so on.

In principle, we can construct a ten-adic number $M = \ldots 3125$ for which there are infinitely many powers of five that end with the same set of digits as M does, for any size set of digits you want. (There are infinitely many powers of five that end with the final set of one-hundred digits as M does, and there are infinitely many powers of five that end with the same million final digits, and so on.)

We can do the same construction for the powers of two and construct a ten-adic number $N = \ldots 832$ for which there exist infinitely many powers of two that end with any final set of digits of N.

Now look what happens when you multiply N and M. You do indeed get zero.

```
              ...   3   1   2   5
           x  ...       8   3   2
         ------------------------
=             ...   6 | 2 | 4 | 10
          ... 9 | 3 | 6 | 15 | 0
   ... 24 | 8 | 16 | 40 | 0 | 0
                ... | 0 | 0 | 0
                              ⋮
         ------------------------
         ... | 48 | 19 | 10  = ... 0|0|0
```

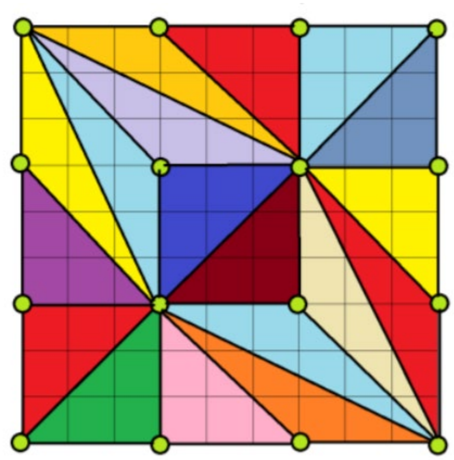
The problem is that 10 is a composite number. One can prove that this annoyance will never arise if one works in base that is a prime number instead!

WHO REALLY CARES ABOUT TEN-ADIC AND OTHER -ADIC SYSTEMS?

If one conducts an internet search on the topic of "p-adic numbers" (here "p" stands for *prime*) one will see the extent to which mathematicians care about these number systems. As I mentioned before, they seem to provide a powerful tool for analyzing multiplicative properties of numbers in basic number theory: the properties of prime numbers and prime factorizations of numbers. So many basic questions about numbers and basic arithmetic are still unanswered!

But perhaps the most surprising application of p-adic numbers I've seen is an application to geometry.

Consider a square. It is possible to subdivide a square into 18 triangles of equal area.



It is fun to find ways to divide a square into 6 triangles of equal area, or 14, or 200. But here's the shocker.

It is impossible to divide a square into an odd number of triangles of equal area!

This result was proved in 1970 by mathematician Paul Monsky ("On Dividing a Square into Triangles". *The American Mathematical Monthly*. **77** (2): 161–164) and he had to use the 2-adic number system to do so!

One can avoid 2-adic numbers if one restricts the vertices of the triangles to grid points (as for the diagram above) and my attempt of explaining the proof in this case is here:

> http://www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_SEPTEMBER-2014_Dividing-Squares-into-Triangles1.pdf.

But I know of no proof of Monsky's result for the general case without using the 2-adic number system.

Mathematics, no matter how abstruse as it might seem, always seems to come up with surprising and unexpected applications.