GLOBAL MATH PROJECT

Uplifting Mathematics for All

Exploding Dots™
Teaching Guide

Experience 6:

All Bases, All at once - Polynomials

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Related resources:

- Access videos of *Exploding Dots™* lessons at: http://gdaymath.com/courses/exploding-dots/
- Be sure to review the *Getting Started* guide, available here.
- Printable student handouts for this experience are available here.

Experience 6: All Bases, All at once - Polynomials Overview

Student Objectives

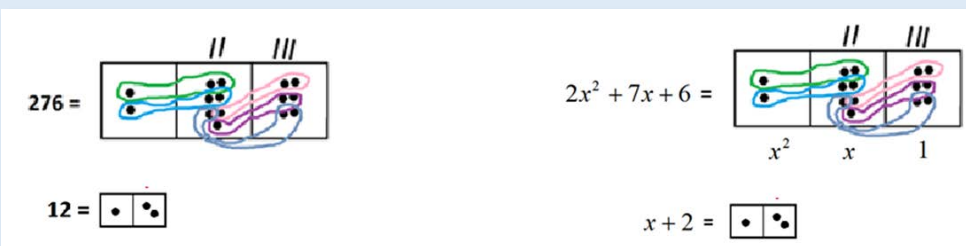
All the work of the previous lessons can be conducted in any base: mathematics does not favor base ten over any other base. Working with unknown base, x, gives the algebra of polynomials; this can be seen as a natural extension of base ten arithmetic.

The Experience in a Nutshell

We humans have a predilection for the number ten, but mathematics does not! All the work of the previous lessons can be conducted in any base.

So, let's be bold and work with all bases, all at once! Let's work with an $1 \leftarrow x$ machine, with x being an unknown value which can be later set to ten, to two, or to any base value one desires.

Objects in an $1 \leftarrow x$ machine are called *polynomials* and the algebra of polynomials is really just a repeat of base ten arithmetic. For instance, look at the grade-school arithmetic problem $276 \div 12$ and the high-school algebra problem $(2x^2 + 7x + 6) \div (x + 2)$. We can see that they are identical, giving the answer 23 and $2x + 3$, respectively. (And if we happen to choose 10 for x, then the latter problem is the first problem!)



Here is an interesting challenge to contend with – allow polynomials with negative coefficients! (This issue tends not to arise in base-ten arithmetic.)

Setting the Scene

View the welcome video from James to set the scene for this experience:
http://gdaymath.com/lessons/explodingdots/6-1-welcome/ [1:12 minutes]

Division in Any Base

This is Core Lesson #13, corresponding to Lesson 6.2 on gdaymath.com/courses/exploding-dots/.

James has a video of this lesson here:

http://gdaymath.com/lessons/explodingdots/6-2-division-base/ [9:59 minutes]

Here is the script James follows when he gives this lesson on a board. Of course, feel free to adapt this wording as suits you best. You will see in the video when and how James draws the diagrams and adds to them.

Okay. Up to now we have been dealing with grade-school or primary-school arithmetic. Let's now head on to advanced high school algebra.

Whoa!

But here's the thing: there is nothing to it. We've already done all the work.

The only thing we have to realize is that there is nothing special about a $1 \leftarrow 10$ machine. We could do all of grade school arithmetic in a $1 \leftarrow 2$ machine if we wanted to, or a $1 \leftarrow 5$ machine, or even a $1 \leftarrow 37$ machine. The math doesn't care in which machine we do it. It is only us humans with a predilection for the number ten that draws us to the $1 \leftarrow 10$ machine.

Let's now go through much of what we've done. But let's now do it in all possible machines, all at once!

Sounds crazy. But it is surprisingly straightforward.

Hopefully you have followed my advice and have kept the picture of $276 \div 12 = 23$ in a $1 \leftarrow 10$ machine on the board. (If not, improvise at the key moment that follows.)

So, what I am going to do is draw a machine on the board, but I am not going to tell you which machine it is. It could be a $1 \leftarrow 10$ machine again, I am just not going to say. Maybe it will be a $1 \leftarrow 2$ machine, or a $1 \leftarrow 4$ machine or a $1 \leftarrow 13$ machine. You just won't know as I am not telling. It's the mood I am in!

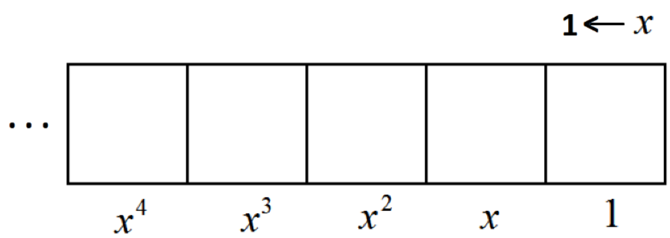
Now, in high school algebra there seems to be a favorite letter of the alphabet to use for a quantity whose value you do not know. It's the letter x.

So, let's work with an $1 \leftarrow x$ machine with the letter x representing some number whose actual value we do not know.

In a $1 \leftarrow 10$ machine the place values of the boxes are the powers of ten: 1, 10, 100, 1000, … .

In a $1 \leftarrow 2$ machine the place values of the boxes are the powers of two: 1, 2, 4, 8, 16, … .

And so on. Thus, in an $1 \leftarrow x$ machine, the place values of the boxes will be the powers of x.



As a check, if I do tell you that x actually is 10 in my mind, then the powers 1, x, x^2, x^3, … match the numbers 1, 10, 100, 1000, … which is correct for a $1 \leftarrow 10$ machine. If, instead, I tell you that x is really 2 in my mind, then the powers 1, x, x^2, x^3, … match the numbers 1, 2, 4, 8, … which is correct for a $1 \leftarrow 2$ machine.

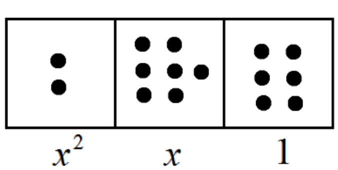
This $1 \leftarrow x$ machine really is representing all machines, all at once!

Okay. Out of the blue! Here's an advanced high school algebra problem.

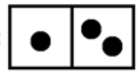
Compute $\left(2x^2+7x+6\right) \div \left(x+2\right)$

I write this problem on the board and insist that the students just do it – despite how scary it might look. There is often some balking, but eventually students get over their nerves and figure out something to do: put two dots in the x^2 box, seven in the x box, and six in the 1 box. Then they tend to draw $x + 2$ as one-dot-next-to-two-dots, and start looking for this pattern in the picture of $2x^2 + 7x + 6$. When the time feels right, I chime in as follows:

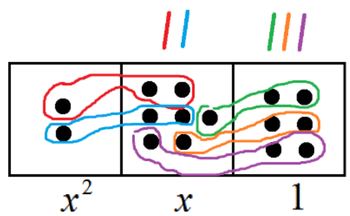
Here's what $2x^2 + 7x + 6$ looks like in an $1 \leftarrow x$ machine. It's two x^2's, seven x's, and six ones.

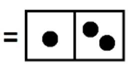
$2x^2 + 7x + 6 =$ 

And here's what $x + 2$ looks like.

$x + 2 =$ 

The division problem $(2x^2 + 7x + 6) \div (x + 2)$ is asking us to find copies of $x + 2$ in the picture of $2x^2 + 7x + 6$.

$2x^2 + 7x + 6 =$ 

$x + 2 =$ 

I see two copies of $x + 2$ at the x level and three copies at the 1 level. The answer is $2x + 3$.

Stare at the picture for $(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$. Does it look familiar?

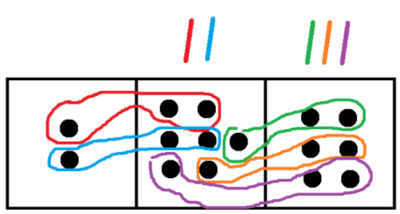
I usually now lean with my back against the wall with one hand pointing to the previous picture of $276 \div 12$ and the other pointing to the new picture of $(2x^2 + 7x + 6) \div (x + 2)$.

We have identical pictures!

We've just done a high school algebra problem as though it is a grade school arithmetic problem!

In a $1 \leftarrow 10$ machine.

$276 \div 12$
$= 23$



In a $1 \leftarrow x$ machine.

$(2x^2 + 7x + 6) \div (x + 2)$
$= 2x + 3$

SAME PICTURE!

What's going on?

Suppose I told you that x really was 10 in my head all along. Then $2x^2 + 7x + 6$ is the number $2 \times 100 + 7 \times 10 + 6$, which is 276. And x + 2 is the number 10 + 2, that is, 12. And so, we computed 276 ÷ 12. We got the answer 2x + 3, which is $2 \times 10 + 3 = 23$, if I am indeed now telling you that x is 10.

So, we did just repeat a grade-school arithmetic problem!

> **Aside:** By the way, if I tell you that x was instead 2, then
>
> $2x^2 + 7x + 6 = 2 \times 4 + 7 \times 2 + 6$, which is 28,
>
> $x + 2 = 2 + 2$, which is 4,
>
> and
>
> $2x + 3 = 2 \times 2 + 3$, which is 7.
>
> We just computed 28 ÷ 4 = 7, which is correct!
>
> Doing division in an $1 \leftarrow x$ machine is really doing an infinite number of division problems all in one hit. Whoa!

Try computing $(2x^3 + 5x^2 + 5x + 6) \div (x + 2)$ in an $1 \leftarrow x$ machine to get the answer $2x^2 + x + 3$ (And if I tell you x is 10 in my mind, can you see that this matches 2256 ÷ 12 = 213?)

Let the students try this.

In high school, numbers expressed in an $1 \leftarrow x$ machine are usually called *polynomials*. They are just like numbers expressed in base 10, except now they are "numbers" expressed in base x. (And if someone tells you x is actually 10, then they really are base 10 numbers!)

Keeping this in mind makes so much of high school algebra so straightforward: it is a repeat of grade school base 10 arithmetic.

I usually have the students try this one too, written in fraction notation: $\frac{x^4+2x^3+4x^2+6x+3}{x^2+3}$.

(The answer is $x^2 + 2x + 1$.)

Handout A: *Division in Any Base*

Use the student handout shown below for students who want practice questions from this lesson to mull on later at home. This is NOT homework; it is entirely optional. (See the document "Experience 6: Handouts" for a printable version.)

Exploding Dots

Experience 6: All Bases, All at once - Polynomials

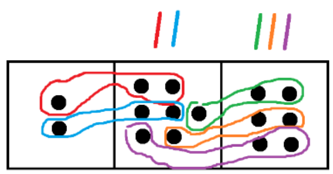
Access videos of all *Exploding Dots* lessons at: http://gdaymath.com/courses/exploding-dots/

Handout A: *Division in Any Base*

The computations $276 \div 12$ and $(2x^2 + 7x + 6) \div (x + 2)$ are identical!

In a $1 \leftarrow 10$ machine.

$$276 \div 12$$
$$= 23$$



In a $1 \leftarrow x$ machine.

$$\left(2x^2 + 7x + 6\right) \div (x + 2)$$
$$= 2x + 3$$

SAME PICTURE!

Here are some practice problems for you to try, if you like:

1. a) Compute $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1)$.

   b) Compute $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1)$.

   If I tell you that x is actually 10 in both these problems what two division problems in ordinary arithmetic have you just computed?

2. Here's a polynomial division problem written in fraction notation. Can you do it? (Is there something tricky to watch out for?)

$$\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$$

3. Show that $(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1)$ equals $x^3 + 3x^2 + 3x + 1$.

   a) What is this saying for $x = 10$?
   b) What is this saying for $x = 2$?
   c) What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11?
   d) What is this saying for $x = 0$?
   e) What is this saying for $x = -1$?

Solutions to Handout A

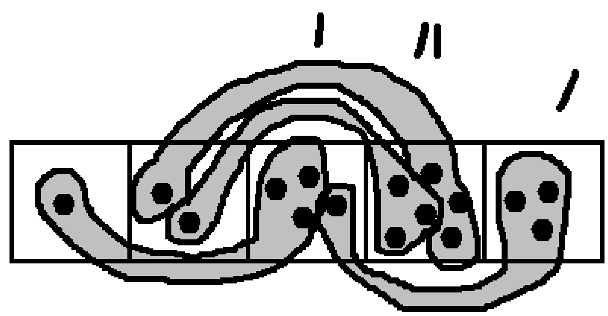
1.

a) $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1) = x^3 + x^2 + 2x + 1$

b) $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1) = x^2 + 2x + 3$

And if x happens to be 10, we've just computed 23541 ÷ 21 = 1121 and 13653 ÷ 111 = 123.

2. We can do it. The answer is $x^2 + 2x + 1$.



3.

a) For $x = 10$ it says 14641 ÷ 11 = 1331

b) For $x = 2$ it says 81 ÷ 3 = 27

c) For $x = 3$ it says 256 ÷ 4 = 64
For $x = 4$ it says 625 ÷ 5 = 125
For $x = 5$ it says 1296 ÷ 6 = 216
For $x = 6$ it says 2401 ÷ 7 = 343
For $x = 7$ it says 4096 ÷ 8 = 512
For $x = 8$ it says 6561 ÷ 9 = 729
For $x = 9$ it says 10000 ÷ 10 = 1000
For $x = 11$ it says 20736 ÷ 12 =1728

d) For $x = 0$ it says 1 ÷ 1 = 1.

e) For $x = -1$ it says 0 ÷ 0 = 0. Hmm! That's fishy! (Can you have a 1 ← 0 machine?)

A Problem!

This is Core Lesson #14, corresponding to Lesson 6.3 on gdaymath.com/courses/exploding-dots/.

James has a video of this lesson here:

http://gdaymath.com/lessons/explodingdots/6-3-problem/ [2:37 minutes]

Okay. Now that we are feeling really good about doing advanced algebra, I have a confession to make. I've been fooling you!

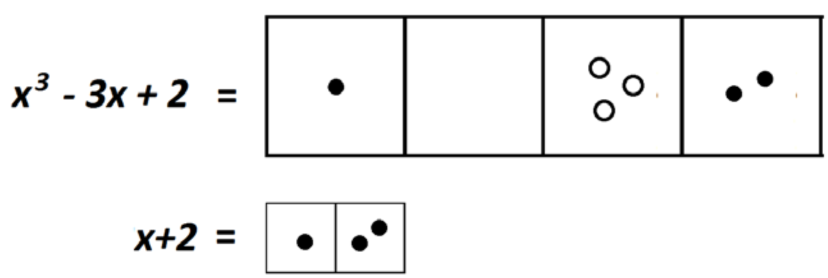
I've been choosing examples that are designed to be nice and to work out just beautifully. The truth is, this fabulous method of ours doesn't usually work so nicely.

Consider, for example,

$$\frac{x^3 - 3x + 2}{x + 2}$$

Do you see what I've been avoiding all this time? Yep. Negative numbers.

Here's what I see in an $1 \leftarrow x$ machine.



We are looking for one dot next to two dots in the picture of $x^3 - 3x + 2$. And I don't see any!

So, what can we do, besides weep a little?

Do you have any ideas?

I really let students struggle with this, insisting that the method really does fail.

At some point, someone will have the idea of unexploding one of the leftmost dots. That is a great idea! It really is. But it has a snag: we don't know the value of x so we don't know how many dots to draw when we unexplode. Bother!

I praise whoever suggested this brilliant idea and commiserate that it just didn't happen to turn out to be helpful.

We need some amazing flash of insight for something clever to do. Or maybe polynomial problems with negative numbers just can't be solved with this dots and boxes method.

So, what do you think? Any flashes of insight?

Resolution

This is Core Lesson #15, corresponding to Lesson 6.4 on gdaymath.com/courses/exploding-dots/.

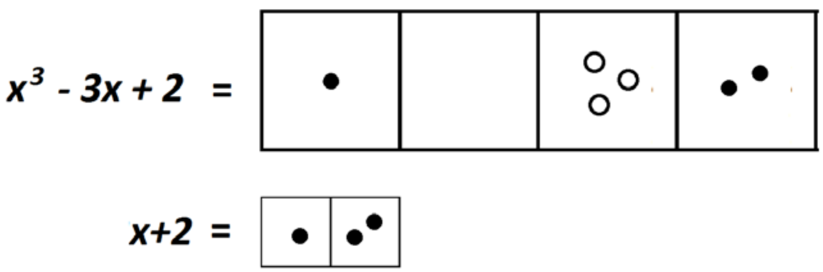
James has a video of this lesson here:

http://gdaymath.com/lessons/explodingdots/6-4-resolution/ [5:55 minutes]

We are stuck on

$$\frac{x^3 - 3x + 2}{x + 2}$$

with the $1 \leftarrow x$ machine picture



We are looking for copies of $x + 2$, one dot next to two dots, anywhere in the picture of $x^3 - 3x + 2$. We don't see any.

And we can't unexplode dots to help us out as we don't know the value of x. (We don't know how many dots to draw when we unexplode.)

The situation seems hopeless at present.

But I have a piece of advice for you, a general life lesson in fact. It's this.

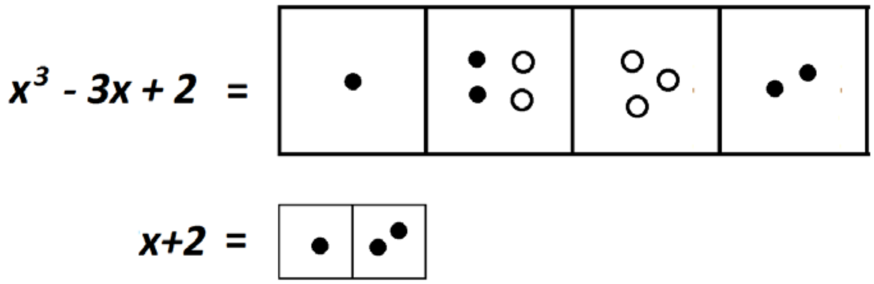
IF THERE IS SOMETHING IN LIFE YOU WANT, MAKE IT HAPPEN! (And deal with the consequences.)

Right now, is there anything in life we want?

Look at that single dot way out to the left. Wouldn't it be nice if we had two dots in the box next to it, to make a copy of $x + 2$?

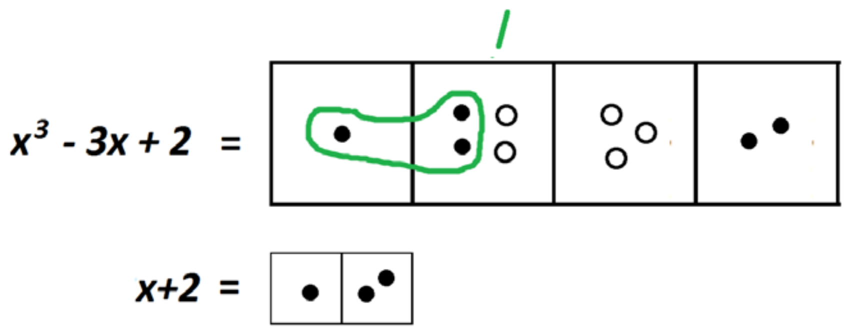
So, let's just put two dots into that empty box! That's what I want, so let's make it happen!

But there are consequences: that box is meant to be empty. And in order to keep it empty, we can put in two antidots as well!



Brilliant!

We now have one copy of what we're looking for.



A student might have suggested adding dots and antidots. In which case, follow their lead, but avoid any specialized jargon they might use (such as a "zero pair"). I rephrase things in simple language.

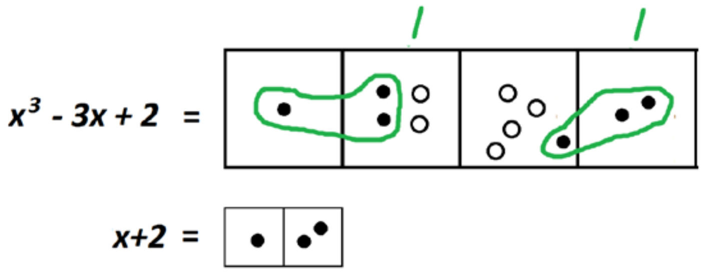
I still talk about the "life lesson" here, saying that "ALEKSANDRA here has hit on a brilliant life lesson."

But there is still the question: Is this brilliant idea actually helpful?

Hmm.

Well. Is there anything else in life you want right now? Can you create another copy of $x + 2$ anywhere?

I'd personally like a dot to the left of the pair dots in the rightmost box. I am going to make it happen! I am going to insert a dot and antidot pair. Doing so finds me another copy of $x + 2$.

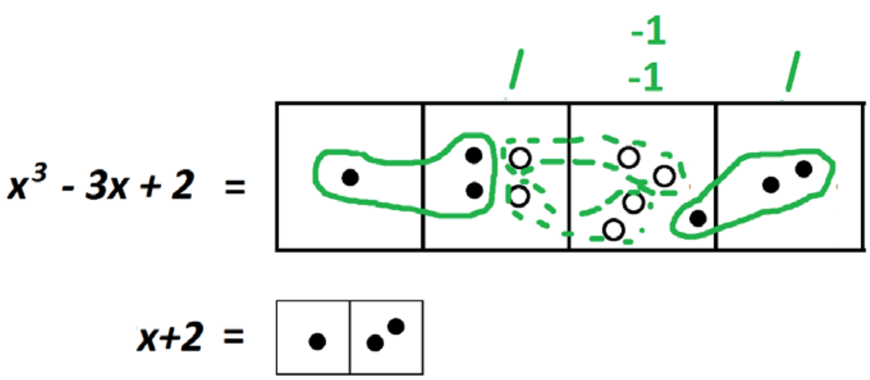


This is all well and good, but are we now stuck? Maybe this brilliant idea really just isn't helpful.

I pause here. A student might suggest what is coming next. If so, follow their lead.

Stare at this picture for a while. Do you notice anything?

Look closely and we start to see copies of the exact opposite of what we're looking for! Instead of one dot next to two dots, there are copies of one antidot next to two antidots.



Whoa!

And how do we read the answer? We see that $(x^3 - 3x + 2) \div (x + 2)$ is $x^2 - 2x + 1$.

Fabulous!

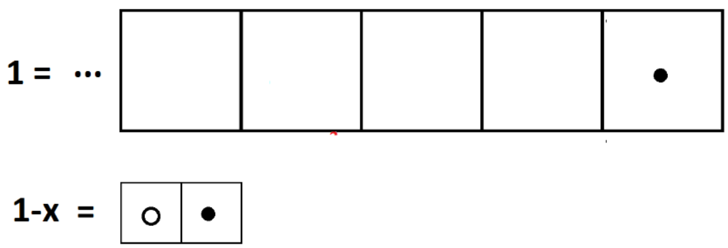
So actually, I was lying about fooling you. We can actually do all polynomial division problems with this dots and boxes method, even ones with negative numbers!

I usually ask students to try one on their own now, such as $\frac{x^{10}-1}{x^2-1}$,

(with answer $x^8 + x^6 + x^4 + x^2 + 1$).

I also usually present some infinite ideas from the next experience right now too. Try presenting this next challenge.

Here is a picture of the very simple polynomial 1 and the polynomial $1 - x$.



Can you compute $\frac{1}{1-x}$? Can you interpret the answer?

One gets the answer $1 + x + x^2 + x^3 + x^4 + \ldots$, an answer that goes on forever. (See the video of Lesson 7.2 here http://gdaymath.com/lessons/explodingdots/7-2-infinite-sums/.) This leads to the famous geometric series formula, usually presented in text books as,

$$1 + x + x^2 + x^3 + x^4 + \cdots \quad = \frac{1}{1-x}$$

$$for - 1 < x < 1$$

I might also give the "optional homework:"

Compute $\frac{1}{1-x-x^2}$ and see if you can discover a very famous sequence of numbers.

Warning: Draw very big boxes!

The answer turns out to be $1 + x + 2x^2 + 3x^2 + 5x^3 + 8x^4 + 13x^5 + 21x^6 + \cdots$ and we see the appearance of the Fibonacci numbers.

Handout B: *A Problem and Resolution*

Use the student handout shown below for students who want practice questions from this lesson to mull on later at home. This is NOT homework; it is entirely optional. (See the document "Experience 6: Handouts" for a printable version.)

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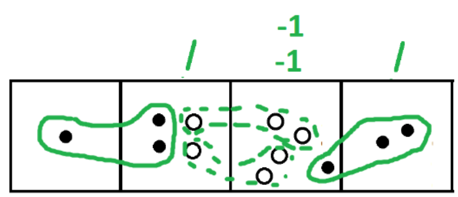
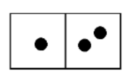
Handout B: *A Problem and Resolution*

We can even work with antidots in polynomial division.

$x^3 - 3x + 2$ =



$x+2$ =



Here are some practice problems for you to try, if you like.

1. Compute $\frac{x^3-3x^2+3x-1}{x-1}$.
2. Try $\frac{4x^3-14x^2+14x-3}{2x-3}$.
3. If you can do this problem, $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1}$, you can probably do any problem!
4. This one is crazy fun: $\frac{x^{10}-1}{x^2-1}$.

Aside: Is there a way to conduct the dots and boxes approach with ease on paper? Rather than draw boxes and dots, can one work with tables of numbers that keep track of coefficients? (The word *synthetic* is often used for algorithms one creates that are a step or two removed from that actual process at hand.)

5. Can you deduce what the answer to $(2x^2+7x+7) \div (x+2)$ is going to be before doing it?

6. Compute $\frac{x^4}{x^2-3}$.

7. Try this crazy one: $\frac{5x^5-2x^4+x^3-x^2+7}{x^3-4x+1}$.

   If you do this with paper and pencil, you will find yourself trying to draw 84 dots at some point. Is it swift and easy just to write the number "84"? In fact, how about just writing numbers and not bother drawing any dots at all?

Solutions to Handout B

1. $\frac{x^3-3x^2+3x-1}{x-1} = x^2-2x+1$.

2. $\frac{4x^3-14x^2+14x-3}{2x-3} = 2x^2-4x+1$.

3. $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1} = 4x^3+2x^2+5x-1$.

4. $\frac{x^{10}-1}{x^2-1} = x^8+x^6+x^4+x^2+1$.

5. We know that $(2x^2+7x+6) \div (x+2) = 2x+3$, so I bet $(2x^2+7x+7) \div (x+2)$ turns out to be $2x+3+\frac{1}{x+2}$. Does it? Can you make sense of remainders?

6. $\frac{x^4}{x^2-3} = x^2+3+\frac{9}{x^2-3}$.

7. $5x^2-2x+21+\frac{-14x^2+86x-14}{x^3-4x+1}$.

Handout C: *Wild Explorations*

Use the student handout shown below for students who want some deep-thinking questions from this Experience to mull on later at home. This is NOT homework; it is entirely optional, but this could be a source for student projects. (See the document "Experience 6: Handouts" for a printable version.)

Exploding Dots

Experience 6: All Bases, All at once - Polynomials

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Handout C: *WILD EXPLORATIONS*

Here are some "big question" investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: CAN WE EXPLAIN AN ARITHMETIC TRICK?

Here's an unusual way to divide by nine.

To compute 21203 ÷ 9, take the digits in "21203" from left to right computing the partial sums along the way as follows

2	**= 2**
2+1	**= 3**
2+1+2	**= 5**
2+1+2+0	**= 5**
2+1+2+0+3	**= 8**

and then read off the answer

$$21203 \div 9 = 2355 \ R\ 8.$$

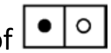
In the same way,

$$1033 \div 9 = 1 \mid 1+0 \mid 1+0+3 \mid \quad R \quad 1+0+3+3 \quad = 114 \ R\ 7$$

and

$$2222 \div 9 = 246 \ R\ 8$$

Can you explain why this trick works?

Here's the approach I might take: for the first example, draw a picture of 21203 in a $1 \leftarrow 10$ machine, but think of nine as $10 - 1$. That is, look for copies of  in the picture.

EXPLORATION 2: CAN WE EXPLORE NUMBER THEORY?

Use an $1 \leftarrow x$ machine to compute each of the following

a) $\frac{x^2-1}{x-1}$ b) $\frac{x^3-1}{x-1}$ c) $\frac{x^6-1}{x-1}$ d) $\frac{x^{10}-1}{x-1}$

Can you now see that $\frac{x^{number}-1}{x-1}$ will always have a nice answer without a remainder?

Another way of saying this is that

$$x^{number} - 1 = (x-1) \times (\text{something}).$$

For example, you might have seen from part c) that $x^6 - 1 = (x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$. This means we can say, for example, that $17^6 - 1$is sure to be a multiple of 16! How? Just choose $x = 17$ in this formula to get

$$17^6 - 1 = (17 - 1) \times (\text{something}) = 16 \times (\text{something}).$$

a) Explain why $999^{100} - 1$ must be a multiple of 998.

b) Can you explain why $2^{100} - 1$ must be a multiple of 3, and a multiple of 15, and a multiple of 31 and a multiple of 1023? (Hint: $2^{100} = (2^2)^{50} = 4^{50}$, and so on.)

c) Is $x^{number} - 1$ always a multiple of $x + 1$? Sometimes, at least?

d) The number $2^{100} + 1$ is not prime. It is a multiple of 17. Can you see how to prove this?