GLOBAL MATH PROJECT

Uplifting Mathematics for All

Exploding Dots™ Teaching Guide

Experience 3:

Addition and Multiplication

Related resources:

- Access videos of *Exploding Dots™* lessons at: http://gdaymath.com/courses/exploding-dots/
- Be sure to review the *Getting Started* guide, available here.
- Printable student handouts for this experience are available here.

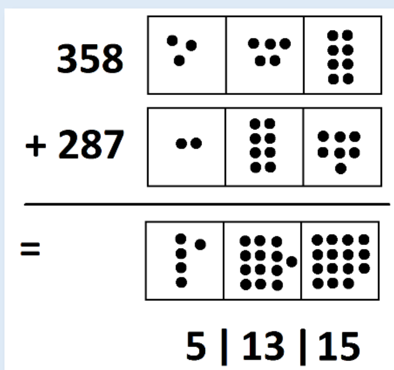
Experience 3: Addition and Multiplication
Overview

Student Objectives

Students now play with the 1 ← 10 machine and examine arithmetic algorithms in the light of the machine. They begin with long addition and then briefly move to multiplication and see the algorithms for them afresh.

The Experience in a Nutshell

To add 358 and 287 simply add together 3 and 2 hundreds, 5 and 8 tens, and 8 and 7 ones. The answer five-hundred thirteenty fifteen results.



This answer is mathematically solid and correct, but it sounds quirky to society. Explosions remedy this to show that this answer is equivalent to 645.

In the same way 26417×3 is 6 | 18 | 12 | 3 | 21. Explosions bring this to an answer society prefers.

Setting the Scene

View the welcome video from James to set the scene for this experience:
http://gdaymath.com/lessons/explodingdots/3-1-welcome/ [0:40 minutes]

Addition

This is Core Lesson # 8, corresponding to Lesson 3.2 on gdaymath.com/courses/exploding-dots/.

James has a video of this lesson here:

http://gdaymath.com/lessons/explodingdots/3-2-addition/ [4:00 minutes]

Here is the script James follows when he gives this lesson on a board. Of course, feel free to adapt this wording as suits you best. You will see in the video when and how James draws the diagrams and adds to them.

Society loves working in base ten. So, let's stay with a $1 \leftarrow 10$ machine for a while and make good sense of all the arithmetic we typically learn in school.

We have just seen how to write numbers. What is the first mathematical thing students learn to do with numbers, once they know how to write them?

Students usually reply "addition" or "to add them."

Okay. Let's explore addition.

Here's an addition problem: Compute $251 + 124$. Such a problem is usually set up this way.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline \end{array}$$

This addition problem is easy to compute: $2 + 1$ is 3, $5 + 2$ is 7, and $1 + 4$ is 5. The answer 375 appears.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline 375 \end{array}$$

But did you notice something curious just then?

Most students notice that I worked from left to right, rather than right to left.

Yes. I worked from left to right just as I was taught to read. I guess this is opposite to what most people are taught to do in a mathematics class: go right to left.

But does it matter? Do you get the same answer 375 if you go right to left instead?

Students say "yes."

So why are we taught to work right to left in mathematics classes?

Many people suggest that the problem we just did is "too nice." We should do a more awkward addition problem, one like $358 + 287$.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

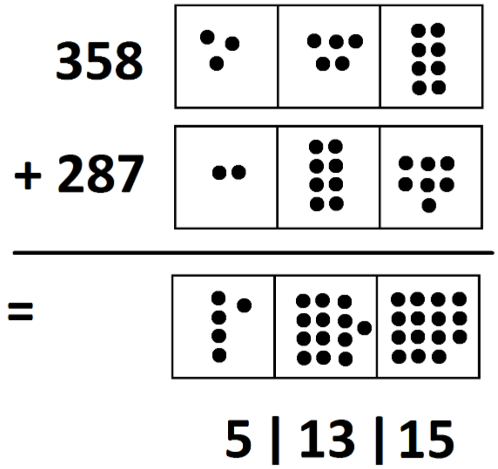
Okay. Let's do it!

If we go from left to right again we get $3 + 2$ is 5; $5 + 8$ is 13; and $8 + 7$ is 15. The answer "five-hundred thirteen-ty fifteen" appears. (Remember, "ty" is short for *ten*.)

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$
$$5\,|\,13\,|\,15$$

I am good at saying "five-hundred thirteenty fifteen" fast and without hesitation. You might want to practice saying it too! Students always laugh at this.

And this answer is absolutely, mathematically correct! You can see it is correct in a 1 ← 10 machine. Here are 358 and 287.



Adding 3 hundreds and 2 hundreds really does give 5 hundreds.

Adding 5 tens and 8 tens really does give 13 tens.

Adding 8 ones and 7 ones really does give 15 ones.

"Five-hundred thirteen-ty fifteen" is absolutely correct as an answer – and I even said it correctly. We really do have 5 hundreds, 13 tens, and 15 ones. There is nothing mathematically wrong with this answer. It just sounds weird. Society prefers us not to say numbers this way.

So, the question is now:

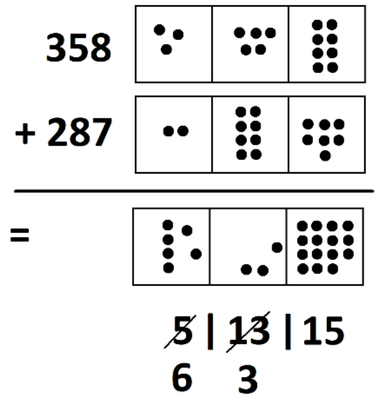
Can we fix up this answer for society's sake – not mathematics' sake – just for society's sake?

The answer is yes! We can do some explosions. (This is a 1 ← 10 machine, after all.)

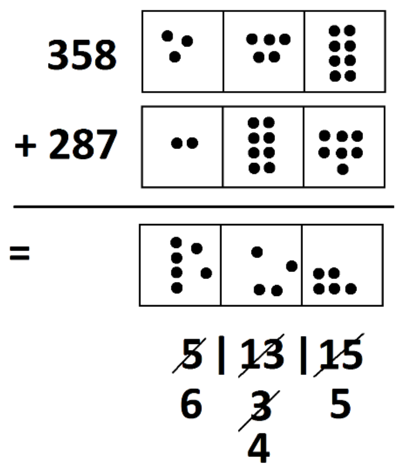
Which do you want to explode first: the 13 or the 15

Most students say the 15. If they do, I say "So you want to go right to left still? Let's do the 13 first then just to break that habit!"

Ten dots in the middle box explode to be replaced by one dot, one place to the left.



The answer “six hundred three-ty fifteen” now appears. This is still a lovely, mathematically correct answer. But society at large might not agree. Let’s do another explosion: ten dots in the rightmost box.



Now we see the answer “six hundred four-ty five,” which is one that society understands. (Although, in English, “four-ty” is usually spelled *forty*.)

Optional Section: *The Traditional Algorithm*

This lesson is not one of the 15 core lessons; it is optional, corresponding to Lesson 3.3 on gdaymath.com/courses/exploding-dots/.

James has a video of this optional lesson here:

http://gdaymath.com/lessons/explodingdots/3-3-optional-traditional-algorithm/.

So how does this dots-and-boxes approach to addition compare to the standard algorithm most people know?

Let's go back to the example $358 + 287$. Most people are surprised (maybe even perturbed) by the straightforward left-to-right answer $5 \mid 13 \mid 15$.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline 5 \mid 13 \mid 15 \end{array}$$

This is because the traditional algorithm has us work from right to left, looking at $8 + 7$ first.

But, in the algorithm we don't write down the answer 15. Instead, we explode ten dots right away and write on paper a 5 in the answer line together with a small 1 tacked on to the middle column. People call this *carrying the one* and it – correctly – corresponds to adding an extra dot in the tens position.

$$\begin{array}{r} {\scriptstyle 1} \\ 358 \\ +\,287 \\ \hline 5 \end{array}$$

Now we attend to the middle boxes. Adding gives 14 dots in the tens box ($5 + 8$ gives thirteen dots, plus the extra dot from the previous explosion).

And we perform another explosion.

$$\begin{array}{r} 1\;1 \\ 358 \\ +\,287 \\ \hline 45 \end{array}$$

On paper, one writes "4" in the tens position of the answer line, with another little "1" placed in the next column over. This matches the idea of the dots-and-boxes picture precisely.

And now we finish the problem by adding the dots in the hundreds position.

$$\begin{array}{r} 1\;1 \\ 358 \\ +\,287 \\ \hline 645 \end{array}$$

So, the traditional algorithm works right to left and does explosions (*"carries"*) as one goes along. On paper, it is swift and compact, and this might be why it has been the favored way of doing long addition for centuries.

The *Exploding Dots* approach works left to right, just as we are taught to read in English, and leaves all the explosions to the end. It is easy to understand and kind of fun.

Both approaches, of course, are good and correct. It is just a matter of taste and personal style which one you choose to do. (And feel free to come up with your own new, and correct, approach too!)

Handout A: *Addition*

Use the student handout shown below for students who want practice questions from this lesson to mull on later at home. This is NOT homework; it is entirely optional. (See the document "Experience 3: Handouts" for a printable version.)

Exploding Dots

Experience 3: Addition and Multiplication

Access videos of all *Exploding Dots* lessons at: http://gdaymath.com/courses/exploding-dots/

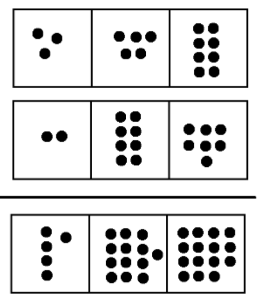
Handout A: *Addition*

Here is the *Exploding Dots* way to add 358 and 287.

358

+ 287

=



5 | 13 | 15

Explosions then show that this answer is equivalent to 645.

Write down the answers to the following addition problems working left to right and not worrying about what society thinks! Then, do some explosions to translate each answer into something society understands.

$$\begin{array}{r} 148 \\ +\ 323 \\ \hline = \end{array} \qquad \begin{array}{r} 567 \\ +\ 271 \\ \hline = \end{array} \qquad \begin{array}{r} 377 \\ +\ 188 \\ \hline = \end{array} \qquad \begin{array}{r} 582 \\ +\ 714 \\ \hline = \end{array}$$

$$\begin{array}{r} 310462872 \\ +\ 389107123 \\ \hline = \end{array} \qquad \begin{array}{r} 87263716381 \\ +\ 18778274824 \\ \hline = \end{array}$$

Solutions to Handout A

$148 + 323 = 4 | 6 | 11 = 471$

$567 + 271 = 7 | 13 | 8 = 838$

$377 + 188 = 4 | 15 | 15 = 5 | 5 | 15 = 565$

$582 + 714 = 12 | 9 | 6 = 1 | 2 | 9 | 6 = 1296$

$310462872 + 389107123 = 6 | 9 | 9 | 5 | 6 | 9 | 9 | 9 | 5 = 699569995$

$$87263716381 + 18778274824 = 9 | 15 | 9 | 13 | 11 | 9 | 8 | 10 | 11 | 10 | 5$$
$$= \ldots = 106041991205$$

Multiplication

This is Core Lesson # 8, corresponding to Lesson 3.4 on gdaymath.com/courses/exploding-dots/.

James has a video of this lesson here:

http://gdaymath.com/lessons/explodingdots/3-4-multiplication/ [2:37 minutes]

Okay. Addition. What do students usually learn to do next in school?

Students invariably respond "subtraction." I respond …

That's too hard. Let's do multiplication instead!

Okay. Multiplication. Let's just do it.

You've got less than three seconds to write down an absolutely, correct speedy answer to this multiplication problem. What's a good answer?

26417 x 3

I usually ham this up a bit. I stand there and count slowly to three or something.

Can you see that 6 | 18 | 12 | 3 | 21, that is, "six ten thousand, eighteen thousand, twelve hundred and three-ty twenty-one," is correct and does the speedy trick?

Here's what's going on.

Let's start with a picture of 26417 in a 1 ← 10 machine. (Is it okay if I just write numbers rather than draw dots?)

2	6	4	1	7

We're being asked to triple this number.

2	6	4	1	7	x 3

Right now, we have 2 ten-thousands. If we triple this, we'd have 6 ten-thousands.

Right now, we have 6 thousands, and tripling would make this 18 thousands.

Also, 4 hundreds becomes 12 hundreds; 1 ten becomes 3 tens; and 7 ones becomes 21 ones.

6	18	12	3	21

We see the answer "sixty eighteen thousand, twelve hundred and three-ty twenty-one." Absolutely solid and mathematically correct!

Now, how can we fix up this answer for society?

Do some explosions of course!

Which explosion do you want to do first?

At this point, students usually pick a middle number rather than the rightmost one. Good!

Okay. Let's explode the 12 first. It gives

6 | 19 | 2 | 3 | 21

Do you want to keep going? Or do you want to just stop there and say we can finish it up if we want to?

Depending on how students respond we either keep going to get the final answer 79251 or we just stop there and move on.

Comment: Students don't usually ask me about long multiplication. If they do, feel free to conduct the remaining two optional sections in this experience.

Handout B: *Multiplication*

Use the student handout shown below for students who want practice questions from this lesson to mull on later at home. This is NOT homework; it is entirely optional. (See the document "Experience 3: Handouts" for a printable version.)

Exploding Dots

Experience 3: Addition and Multiplication

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Handout B: *Multiplication*

We see that

$$26417 \times 3 = 6 \,|\, 18 \,|\, 12 \,|\, 3 \,|\, 21$$

2	6	4	1	7	x 3	=	6	18	12	3	21

With explosions, this answer can be rewritten 79251.

Here are some more questions you might or might not choose to ponder.

Compute each of the following: 26417×4, 26417×5, and 26417×9.

Compute 26417×10 and explain why the answer has to be 264170.

(This answer looks like the original number with the digit zero tacked on to its end.)

Extra: Care to compute 26417×11 and 26417×12 too?

(The answer could be, "No! I do not care to do this!)

Solutions to Handout B

We have

$$26417 \times 4 = 8|24|16|4|28 = 10|4|16|4|28 = 1|0|4|16|4|28 = 1|0|5|6|4|28 = 105668$$
$$26417 \times 5 = 10|30|20|5|35 = 10|30|20|8|5 = 10|32|0|8|5 = 13|2|0|8|5 = 132085$$
$$26417 \times 9 = 18|54|36|9|63 = 18|54|36|15|3 = \ldots = 237753$$

$$26417 \times 10 = 20|60|40|10|70 = \cdots = 264170$$

and

$$26417 \times 11 = 22|66|44|11|77 = \ldots = 290587$$

$$26417 \times 12 = 24|72|48|12|84 = \ldots = 317004$$

For a full discussion as to why 26417×10 is 264170 have a look at the final section of this experience.

Optional Section: *Multiplying by Ten*

This lesson is not one of the 15 core lessons; it is optional, corresponding to Lesson 3.5 on gdaymath.com/courses/exploding-dots/.

Why must the answer to 26417×10 *look like the original number with a zero tacked on to its end?*

I remember being taught this rule in school: to multiply by ten tack on a zero. For example,

$$37 \times 10 = 370$$

$$98989 \times 10 = 989890$$

$$100000 \times 10 = 1000000$$

and so on.

This observation makes perfect sense in the dots-and-boxes thinking.

Here's the number 26417 again in a $1 \leftarrow 10$ machine.

2	6	4	1	7

Here's 26417×10.

20	60	40	10	70

Now let's perform the explosions, one at a time. (We'll need an extra box to the left.)

We have that 2 groups of ten explode to give 2 dots one place to the left, and 6 groups of ten explode to give 6 dots one place to the left, and 4 groups of ten explode to give 4 dots one place to the left, and so on. The digits we work with stay the same. In fact, the net effect of what we see is all digits shifting one place to the left to leave zero dots in the ones place.

	20	60	40	10	70

2	0	60	40	10	70

2	6	0	40	10	70

2	6	4	0	10	70

2	6	4	1	0	70

2	6	4	1	7	0

Indeed, it looks like we just tacked on a zero to the right end of 26417. (But this is really because of a whole lot of explosions.)

Here are two practice problems:

a) What must be the answer to 476×10? To 476×100?

b) What must be the answer to $9190 \div 10$? To $3310000 \div 100$?

Optional Section: *Long Multiplication*

This lesson is not one of the 15 core lessons; it is optional, corresponding to Lesson 3.6 on gdaymath.com/courses/exploding-dots/.

Is it possible to do, say, 37×23, with dots and boxes?

Here we are being asked to multiply three tens by 23 and seven ones by 23. If you are good with your multiples of 23, this must give $3 \times 23 = 69$ tens and $7 \times 23 = 161$ ones. The answer is thus $69|161$. With explosions, this becomes 851.

But this approach seems hard! It requires you to know multiples of 23.

Thinking Exercise:

Suzzy thought about 37×23 for a little while, she eventually drew the following diagram.

	6	14	0
+		9	21
=	6	23	21

She then said that $37 \times 23 = 6|23|21 = 8|3|21 = 851$.

a) Can you work out what Suzzy was thinking? Here's another example she later did.

				3	1	2	5
			x		8	3	2
=				6	2	4	10
			9	3	6	15	0
		24	8	16	40	0	0
		24	17	25	48	19	10
=	2	6	0	0	0	0	0

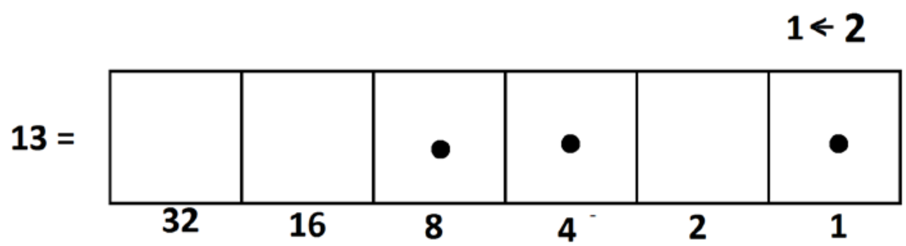
b) What diagram do you think Suzzy might draw for 236×34 (and what answer will she get from it)?

c) Using Suzzy's approach do 37×23 and 23×37 give the same answer? Is it obvious as you go through the process that they will? Do 236×34 and 34×236 give the same answer in Suzzy's approach?

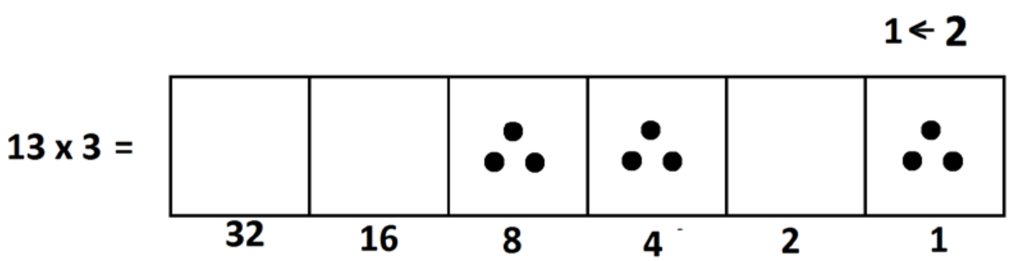
Here's another fun way to think about multiplication. Let's work with a $1 \leftarrow 2$ machine this time.

Let's compute 13×3.

Here's what 13 looks like in a $1 \leftarrow 2$ machine.

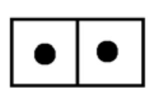


We're being asked to triple everything. So, each dot we see is to be replaced by three dots.

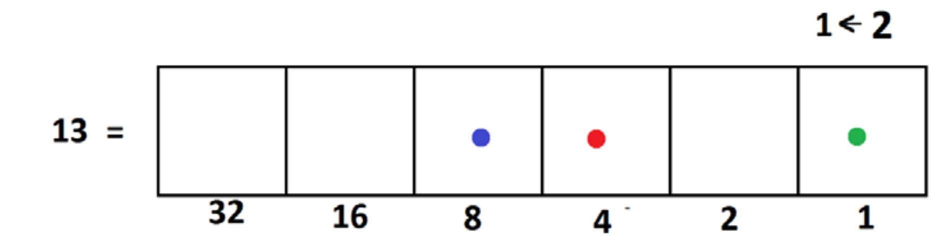


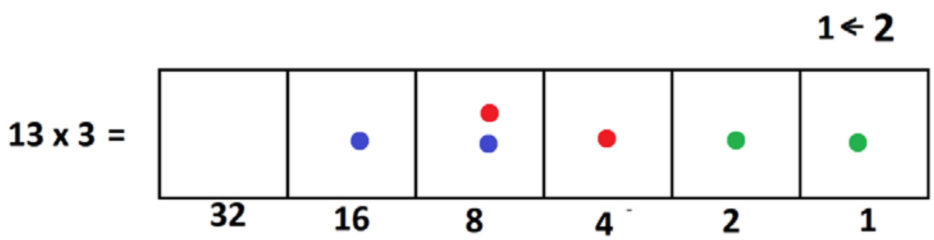
And now we can do some explosions to see the answer 39 appear (which is 100111 in the $1 \leftarrow 2$ machine).

Alternatively, we can notice that three dots in a $1 \leftarrow 2$ machine actually look like this.



So, we can replace each dot in our picture of 13 instead by one dot and a second dot one place to the left. (I've added some color to the picture to help.)

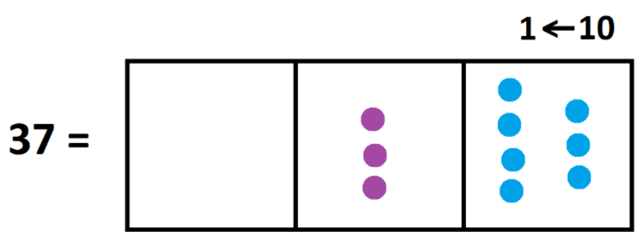




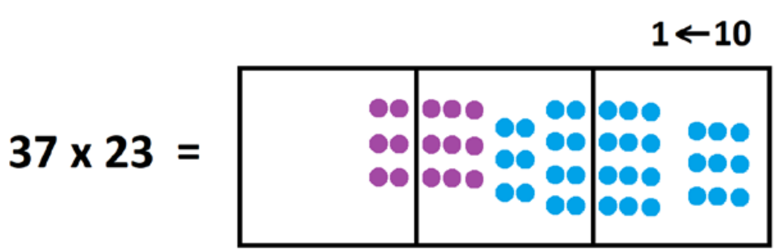
Now with fewer explosions to do, we see the answer 100111 appear.

We can follow this latter approach in base ten too, if we like, but it its likely to be unpleasant and messy! (I personally like Suzzy's approach in the Thinking Exercise above.)

Let's consider 37×23 again. Here's what 37 looks like.



To multiply by 23, we need to replace each single dot with twenty-three dots. But since 23 looks like two-dots-next-to-three dots in a $1 \leftarrow 10$ machine, we can replace each dot with two dots and three dots.



We see the answer $6|21|23$, which explodes to give the answer 851.

Handout C: *Wild Explorations*

Use the student handout shown below for students who want some deep-thinking questions from this Experience to mull on later at home. This is NOT homework; it is entirely optional, but this could be a source for student projects. (See the document "Experience 3: Handouts" for a printable version.)

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Handout C: *WILD EXPLORATIONS*

Here are some "big question" investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: THERE IS NOTHING SPECIAL ABOUT BASE TEN FOR ADDITION

Here is an addition problem in a $1 \leftarrow 5$ machine. (That is, it is a problem in base five.) This is not a $1 \leftarrow 10$ machine addition.

$$\begin{array}{r} 20413 \\ +\ 13244 \\ \hline \end{array}$$

a) What is the $1 \leftarrow 5$ machine answer?

b) What number has code 20413 in a $1 \leftarrow 5$ machine? What number has code 13244 in a $1 \leftarrow 5$ machine? What is the sum of those two numbers and what is the code for that sum in a $1 \leftarrow 5$ machine?

[Here are the answers so that you can check your clever thinking.

The sum, as a $1 \leftarrow 5$ machine problem, is

$$20413+13244=3|3|6|5|7=3|4|1|5|7=3|4|2|0|7=3|4|2|1|2=34212.$$

In a $1 \leftarrow 5$ machine, 20413 is two 625's, four 25's, one 5, and three 1's, and so is the number 1358 in base ten; 13244 is the number 1074 in base ten; and 34212 is the number 2432 in base ten. We have just worked out $1358+1074=2432$.]

EXPLORATION 2: THERE IS NOTHING SPECIAL ABOUT BASE TEN FOR MULTIPLICATION

Let's work with a $1 \leftarrow 3$ machine.

a) Find 111×3 as a base three problem. Also, what are 1202×3 and 2002×3?

Can you explain what you notice?

Comment: For base three, we could write "10" here instead of "3".

Let's now work with a $1 \leftarrow 4$ machine.

b) What is 133×4 as a base four problem? What is 2011×4? What is 22×4?

Can you explain what you notice?

Comment: For base four, we could write "10" here instead of "4".

In general, if we are working with a $1 \leftarrow b$ machine, can you explain why multiplying a number in base b by b returns the original number with a zero tacked on to its right?