Exploding Dots™
Teaching Guide

Experience 2:
Insight

Related resources:

- Access videos of *Exploding Dots™* lessons at: http://gdaymath.com/courses/exploding-dots/
- Be sure to review the *Getting Started* guide, available here.
- Printable student handouts for this experience are available here.

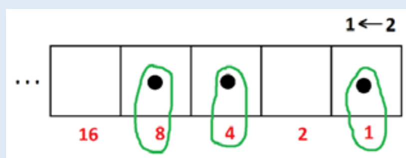
Experience 2: Insight
Overview

Student Objectives

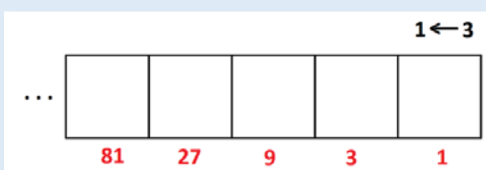
Students discover that the dots-and-boxes machine codes are the representations of numbers in different bases. The 1 ← 2 machine codes are the binary (base two) representations of numbers, the 1 ← 3 machine codes are the ternary (base three) representations of numbers, and the 1 ← 10 machine codes are our familiar base-ten representations.

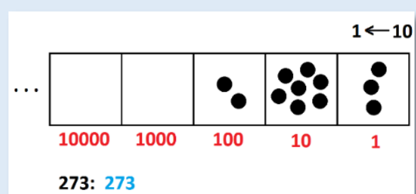
The Experience in a Nutshell

In a 1 ← 2 machine a pair of dots in any one box is equivalent to a single dot, one box to their left. As dots in the rightmost box are set to be worth 1, the value of each dot in subsequent boxes must arise from a process of doubling. We can now readily see that the 1 ← 2 machine code 1101 for the number thirteen, for instance, is correct: thirteen is one 8, one 4, and one 1.



The 1 ← 3 machine gives the ternary representations of numbers. The 1 ←10 machine gives our familiar base-ten number representations.





We even speak the language of base-ten representations.

273 = two hundred seventy three

Setting the Scene

View the welcome video from James to set the scene for this experience:
http://gdaymath.com/lessons/explodingdots/2-1-welcome/ [0:38 minutes]

Explaining the 1 ← 2 Machine

This is Core Lesson # 4, corresponding to Lesson 2.2 on gdaymath.com/courses/exploding-dots/.

The three core lessons of this experience are very swift. James has a video that covers all three together here:

> http://gdaymath.com/lessons/explodingdots/2-2-explaining-1-leftarrow-2-machine/ [8:42 minutes]

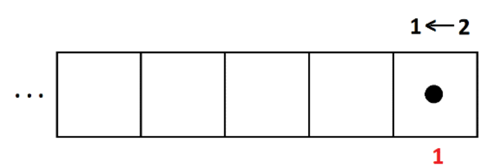
Here is the script James follows when he gives this lesson on a board. Of course, feel free to adapt this wording as suits you best. You will see in the video when and how James draws the diagrams and adds to them.

All right. It's time to explain what the machines are really doing. (Have you already figured it all out?)

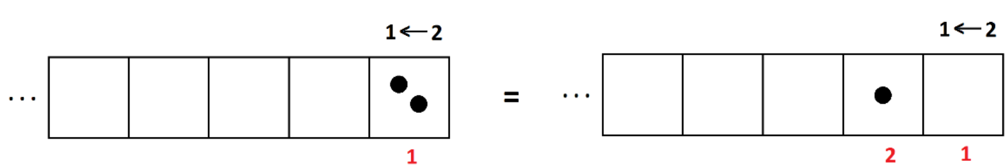
Let's go back to the 1 ← 2 machine and first make sense of that curious device. Recall that it follows the rule:

> *Whenever there are two dots in any one box they "explode," that is, disappear, and are replaced by one dot, one place to their left.*

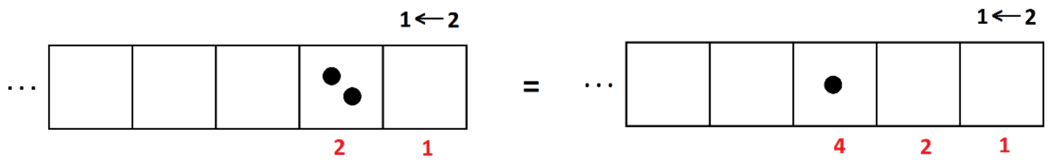
And this machine is set up so that dots in the rightmost box are always worth one.



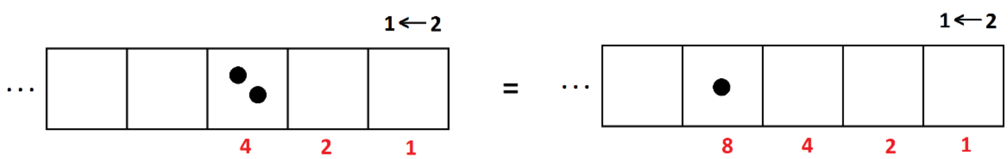
With an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1's, that is, 2.



And two dots in this second box is equivalent to one dot, one place to the left. Such a dot must be worth two 2's, that is, worth 4.



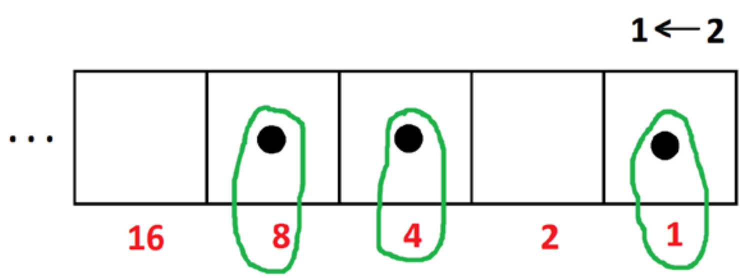
And two 4's makes 8 for the value of a dot the next box over.



And two 8's make 16, and two 16's make 32, and two 32's make 64, and so on.

Students sometimes like to keep listing the doubling numbers. Have fun with them on this if they do. (I usually pretend to get bored after a while so that the game does eventually end!)

We saw earlier that the code for thirteen in a $1 \leftarrow 2$ machine is 1101. Now we can see that this is absolutely correct: one 8 and one 4 and no 2's and one 1 does indeed make thirteen.



People call the $1 \leftarrow 2$ codes for numbers the *binary* representations of numbers (with the prefix *bi-* meaning "two"). They are also called *base two* representations. One only ever uses the two symbols 0 and 1 when writing numbers in binary.

Computers are built on electrical switches that are either on, or off. So, it is very natural in computer science to encode all arithmetic in a code that uses only two symbols: say 1 for "on" and 0 for "off." Thus, base two, binary, is the right base to use in computer science.

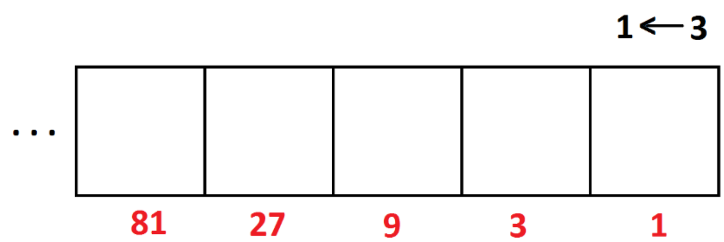
Explaining More Machines

This is Core Lesson # 5, corresponding to Lesson 2.3 on gdaymath.com/courses/exploding-dots/.

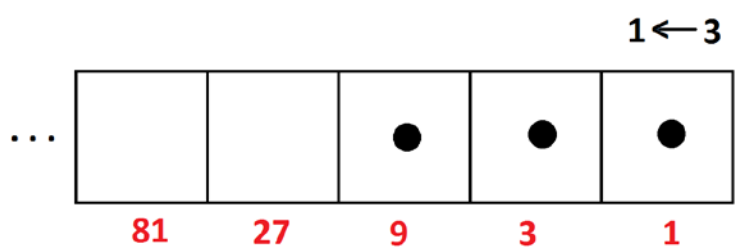
Again, the video that covers Core Lessons #4-6 is here:

http://gdaymath.com/lessons/explodingdots/2-2-explaining-1-leftarrow-2-machine/ [8:42 minutes]

In a $1 \leftarrow 3$ machine, three dots in any one box are equivalent to one dot, one place to the left. (And each dot in the rightmost box is again worth 1.) We get the dot values in this machine by noting that three 1's is 3, and three 3's is 9, and three 9's is 27, and so on.



At one point, we said that the $1 \leftarrow 3$ code for thirteen is 111. And we see that this is correct: one 9 and one 3 and one 1 does indeed make thirteen.



The $1 \leftarrow 3$ machine codes for numbers are called *ternary* or *base three* representations of numbers. Only the three symbols 0, 1, and 2 are ever needed to represent numbers in this system.

Optionally, share the following.

> Scientists are discussing the idea of building optic computers based on polarized light: either light travels in one plane, or in a perpendicular plane, or there is no light. For these computers, base three arithmetic would be the appropriate notational system to use.

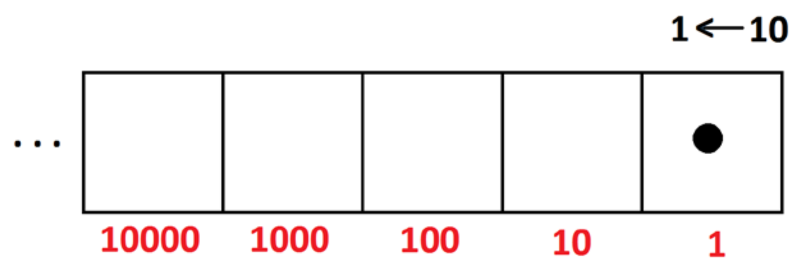
We Speak 1 ← 10 Machine

This is Core Lesson # 6, corresponding to Lesson 2.4 on gdaymath.com/courses/exploding-dots/.

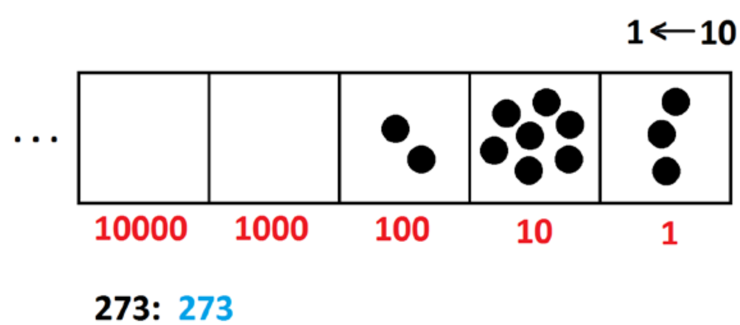
Again, the video that covers Core Lessons #4-6 is here:

http://gdaymath.com/lessons/explodingdots/2-2-explaining-1-leftarrow-2-machine/ [8:42 minutes]

And finally, for a 1 ← 10 machine, we see that ten ones makes 10, ten tens makes 100, ten one-hundreds makes 1000, and so on. A 1 ← 10 has ones, tens, hundreds, thousands, and so on, as dot values.



We saw that the code for the number 273 in a 1 ← 10 machine is 273, and this is absolutely correct: 273 is two hundreds, seven tens, and three ones.



In fact, we even speak the language of a 1 ← 10 machine. When we write 273 in words, we write

273 = two hundred seventy three

We literally say, in English at least, two HUNDREDS and seven TENS (that "ty" is short for "ten") and three.

So, through this untrue story of dots and boxes we have discovered *place-value* and *number bases*: base two, base three, base ten, and so on. And society has decided to speak the language of base ten machine.

Why do you think we humans have a predilection for the 1 $\leftarrow$ 10 machine? Why do we like the number ten for counting?

One answer could be because of our human physiology: we are born with ten digits on our hands. Many historians do believe this could well be the reason why we humans have favored base ten.

There are some cultures on this planet that have used base twenty. Why might they have chosen that number do you think?

In fact, there are vestiges of base twenty thinking in western European culture of today. For example, in French, the number 87 is spoken and written as *quatre-vingt-sept*, which translates, word for word, as "four twenties seven." In the U.S. the famous Gettysburg address begins: "Four score and seven years ago." That's four-twenties and seven years ago.

I happen to know that Martians have four fingers on each of two hands. What base do you think they might use in their society? Probably base eight.

All right. The point of this lesson has been made. We have discovered base-ten place value for writing numbers and seen their context in the whole story of place value. We humans happen to like base-ten in particular, because that is the number of fingers most of us seem to have.

Handout A: *Explaining the Machines*

Use the student handout shown below for students who want practice questions from this lesson to mull on later at home. This is NOT homework; it is entirely optional. (See the document "Experience 2: Handouts" for a printable version.)

Exploding Dots

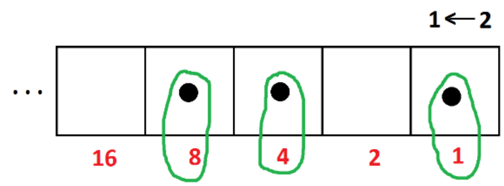
Experience 2: Insight

Access videos of all *Exploding Dots* lessons at: http://gdaymath.com/courses/exploding-dots/

Handout A: *Explaining the Machines*

In a $1 \leftarrow 2$ machine a pair of dots in any one box is equivalent to a single dot, one box to their left. As dots in the rightmost box are worth 1, dots in subsequent boxes are thus worth 2, 4, 8, 16, and so on.

We can see that the $1 \leftarrow 2$ machine code 1101 for the number thirteen, for instance, is correct: thirteen is one 8, one 4, and one 1.

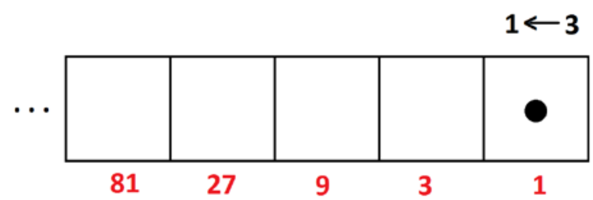


Here are some questions you might, or might not, want to try:

1. What number has $1 \leftarrow 2$ machine code 100101?

2. What is the $1 \leftarrow 2$ machine code for the number two hundred?

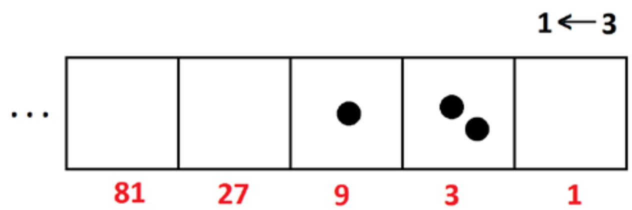
In a $1 \leftarrow 3$ machine, three dots in any one box are equivalent to one dot one place to the left. (And each dot in the rightmost box is again worth 1.) We get the dot values in this machine by noting that three 1's is 3, and three 3's is 9, and three 9's is 27, and so on.



3.

a) What is the value of a dot in the next box to the left after the ones shown?

b) The 1 ← 3 machine code for fifteen is 120. We see that this is correct as one 9 and two 3's does indeed make fifteen.



Could we also say that the 1 ← 3 code for fifteen is 0120? That is, is it okay to put zeros in the front of these codes? What about zeros at the ends of codes? Are they optional?

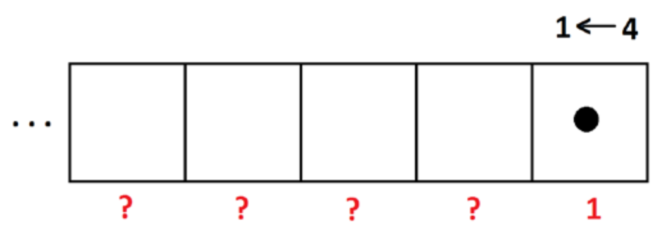
Is it okay to leave off the last zero of the code 120 for fifteen and just write instead 12?

c) What number has 1 ← 3 machine code 21002?

d) What is the 1 ← 3 machine code for two hundred?

4.

a) In the 1 ← 4 system four dots in any one box are equivalent to one dot, one place to their left. What is the value of a dot in each box?



b) What is the 1 ← 4 machine code for twenty-nine?

c) What number has 132 as its 1 ← 4 machine code?

5. I happen to know that *Venutians* have six fingers on each of two hands. What base do you think they might use in their society?

Solutions to Handout A

1. Thirty-seven. It’s a 32 and a 4 and a 1.

2. 11001000

3.

a) Each dot in the next box to the left is worth three 81’s, that’s 243.

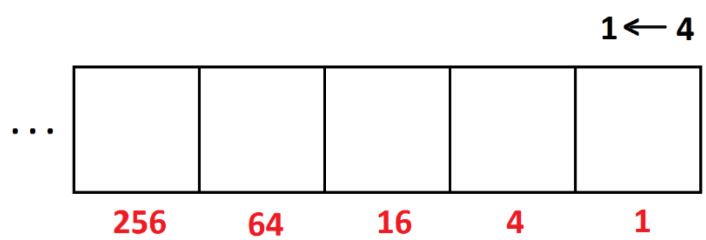
b) Yes it is okay to insert a zero at the front of the code. This would say that there are no 27’s, which is absolutely correct. Deleting the end zero at the right, however, is problematic. 120 is the code for fifteen (one 9 and two 3’s) but 12 is the code for five (one 3 and two 1’s).

c) One hundred and ninety one. (Two 81’s, one 27, and two 1’s.)

d) 21102

4.

a) For a 1 ← 4 machine, boxes have the following values:



b) The number twenty-nine has code 131 in a 1 ← 4 machine.

c) Thirty. (This is one more than the code for twenty-nine!)

5. Might Venutians use base twelve? This means they will need twelve different symbols for writing numbers.
By the way, have you noticed that we use ten different symbols – 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 – which we call *digits*. (We call our fingers *digits* too!)

Handout B: *Wild Explorations*

Use the student handout shown below for students who want some deep-thinking questions from this Experience to mull on later at home. This is NOT homework; it is entirely optional, but this could be a source for student projects. (See the document "Experience 2: Handouts" for a printable version.)

Exploding Dots

Experience 2: Insight

Access videos of all *Exploding Dots* lessons at: http://gdaymath.com/courses/exploding-dots/

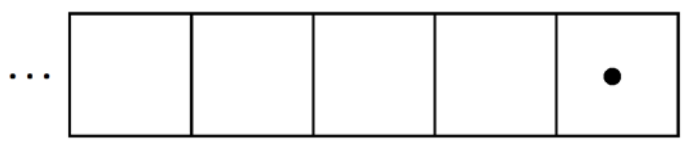
Handout B: *WILD EXPLORATIONS*

Here are some "big question" investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: CAN MACHINES "GO THE OTHER WAY"?

Jay decides to play with a machine that follows a $1 \leftarrow 1$ rule. He puts one dot into the right-most box. What happens? Do assume there are infinitely many boxes to the left.

$1 \leftarrow 1$



Suggi plays with a machine following the rule $2 \leftarrow 1$. She puts one dot into the right-most box. What happens for her?

Do you think these machines are interesting? Is there much to study about them?

EXPLORATION 2: CAN WE PLAY WITH WEIRD MACHINES?

Poindexter decides to play with a machine that follows the rule $2 \leftarrow 3$.

a) Describe what happens when there are three dots in a box.
b) Work out the $2 \leftarrow 3$ machine codes for the numbers 1 up to 30. Any patterns?
c) The code for ten in this machine turns out to be 2101. Look at your code for twenty. Can you see it as the answer to "ten plus ten"? Does your code for thirty look like the answer to "ten plus ten plus ten"?

Comment: We'll explore this weird $2 \leftarrow 3$ machine in Experience 9. It is mighty weird!