Are All U-Shaped Graphs Quadratic?

Teaching Guide

UPPER MIDDLE SCHOOL/HIGH SCHOOL

Video Resource

Access videos of *Garden Paths* lessons at [https://gdaymath.com/courses/gmp/.](https://gdaymath.com/courses/gmp/)

Student Handouts

All practice problems, and solutions, in an accompanying document.

Are All U-Shaped Graphs Quadratic?: Overview

Student Objectives

Students develop, with sound understanding, numerical techniques for determine whether or not a set of data or a given graph of a curve is quadratic. If there is an *a priori* reason to believe a particular data set follows a certain pattern, students can also use those techniques to find a formula for that data.

The Experience in a Nutshell

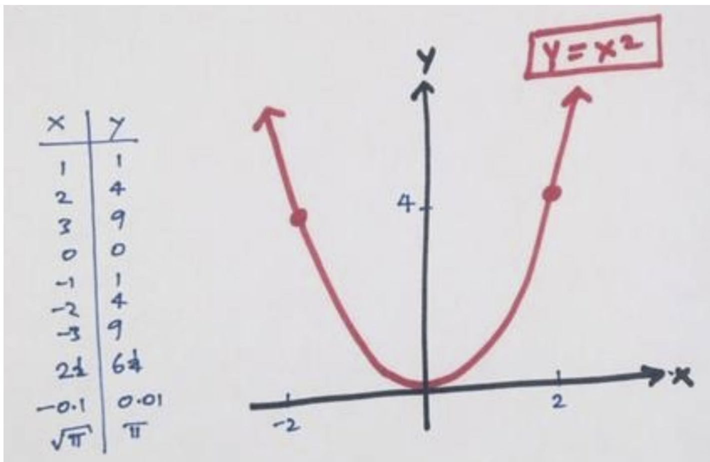
Students are often led to believe that a goal in mathematics work is to find a pattern with the implicit message that patterns should be trusted. Algebra students are also implicitly led to believe that curves that look like they could be quadratic are quadratic. After all, the only U-shape curves studied in a standard algebra curriculum are quadratic curves and examples of non-quadratic curve are never displayed.

In this experience we bring into question whether all graphs of curve that look "U-shaped" actually are quadratic. We explore some historic roots to this study through the work of Galileo, before embarking on a robust study of integer sequences and the consequence of constant-differences patterns they seem to follow. If the patterns can be trusted, we present a technique for finding formulas for these sequences.

An aside explores the ancient Greek definition of a *parabola* and links that notion to the non-obvious claim often implicitly made that graphs of quadratic curves are parabolas.

Lesson 1: Some U-Shaped Graphs

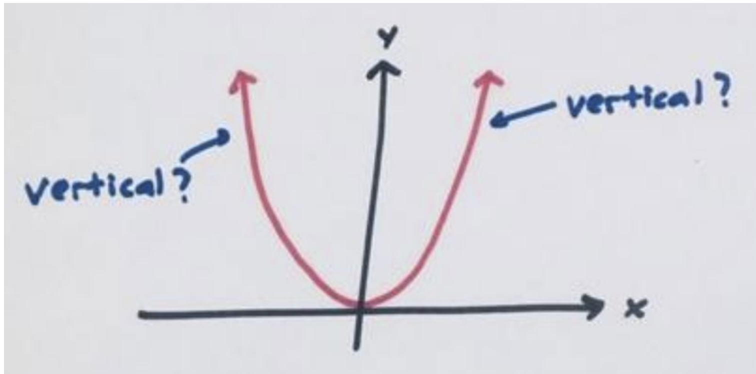
One of the first—usually the first—non-straight-line graph one encounters in school is the graph of a quadratic equation. One starts by drawing the graph associated with the quadratic equation $y = x^2$ to see a beautiful symmetrical U-shaped curve.



Just to be clear, the *graph of an equation* is a plot of all the data points that make the given equation a true sentence about numbers. For example, choosing $x = 2$ and $y = 4$ makes $y = x^2$ a true number sentence, but choosing $x = 3$ and $y = 7$ does not. So we plot the point $(2,4)$ as part of the graph of $y = x^2$, but not the point $(3,7)$.

And the graph of $y = x^2$ is indeed beautifully symmetric. The points $(-2,4)$ and $(2,4)$ both appear on the graph, as do the points $(-3,9)$ and $(3,9)$, and $(-\sqrt{\pi},\pi)$ and $(\sqrt{\pi},\pi)$, for instance. We have a U-shaped graph with a vertical line of symmetry.

Practice 1: *Is the term "U-shaped" actually correct for the shape of the $y = x^2$ graph? After all, the sides of the letter U are vertical. Is the graph of $y = x^2$ ever vertical?*

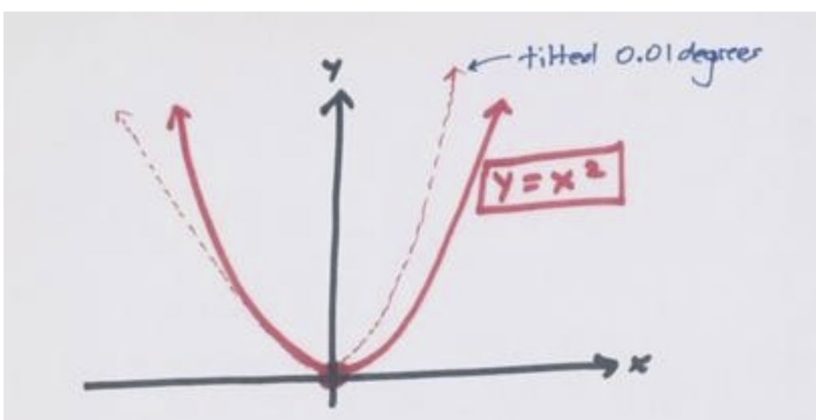


(All solutions to practice questions appear at the end of this guide.)

I'll continue to use the phrase *U-shaped curve* or *U-shaped graph* even if it is technically not correct. Let's simply understand that I am referring to the shape of a quadratic equation graph.

These U-shaped graphs are full of surprises. Here's one.

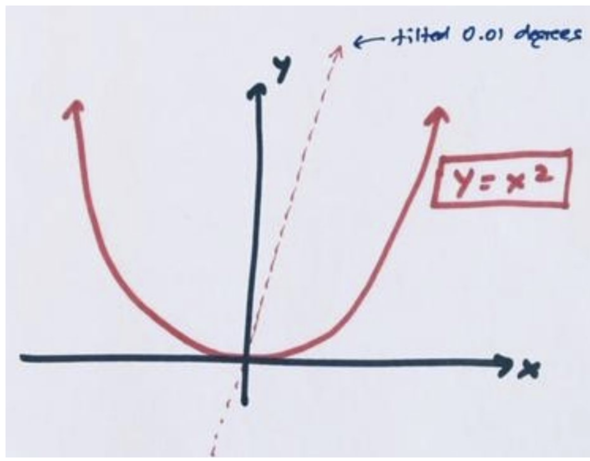
Practice 2: *Suppose we tilt the graph of $y = x^2$ (well, rotate it, actually) counterclockwise about the origin just 0.01°. Does the y-axis intercept this tilted graph at some large non-zero value?*



Again, the solutions to all these practice problems appear at the end of this guide. But let me help you out with this question right now as it seems extraordinarily tough to answer.

Mathematicians are also human and get scared by questions and math challenges too. But one technique they often employ when faced with a question they don't first know how to answer is to change the question!

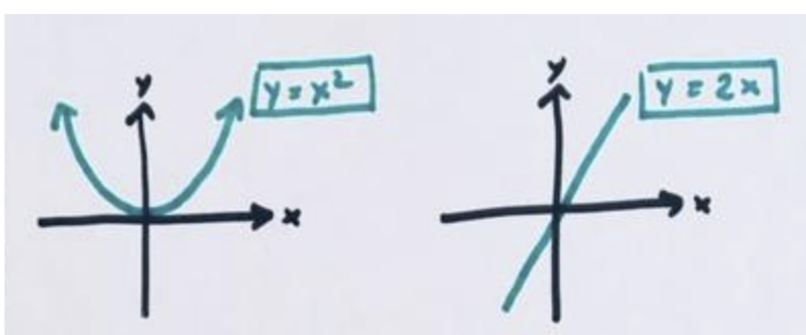
What's the scariest part of this question? It's tilting the U-shaped graph. How does one actually do that? I don't personally know! So, let's tilt something else instead. What if we keep the graph of $y = x^2$ the same and tilt the vertical axis 0.01° clockwise instead? Does this titled line intersect the U-shape graph?



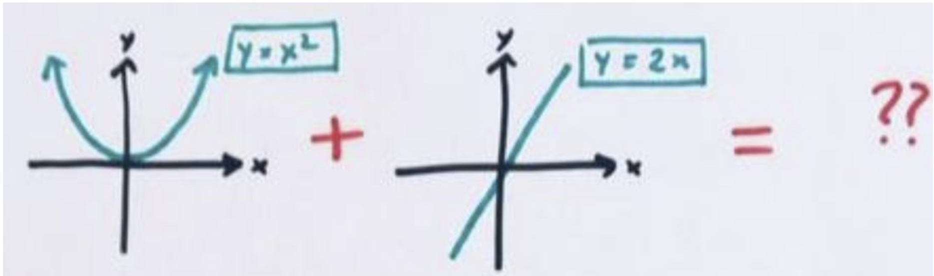
Oh! Hang on! Is this equivalent to answering the original problem?

Here's the biggest surprise of these U-shaped graphs of all. Many people seem to miss this shocker when they first study quadratic equations.

Practice 3: *The graph of $y = x^2$ is a symmetrical U-shaped curve. The graph of $y = 2x$, on the other hand, is not! It is a straight line through the origin with anti-symmetry if you like—the data plot rises to the right as it decreases to the left.*



Now for a weird question: What picture would be obtain if we "added" these two graphs?



What could we mean by this?

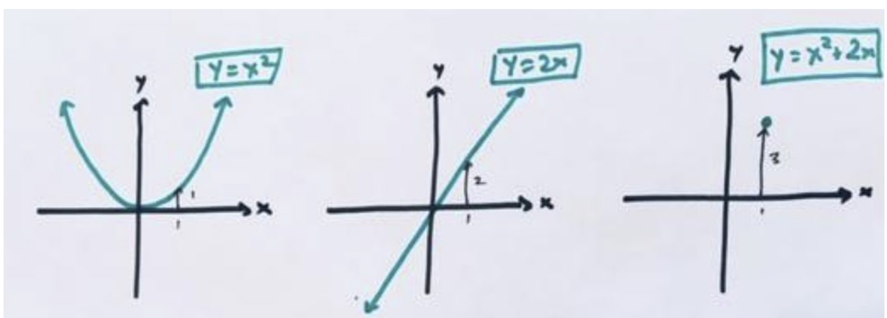
Let's look at each x *value and add their matching* y *values. For example:*

For $x=1$*, we'll add* $(1)^2$ *and* $2(1)$ *and get* 3.

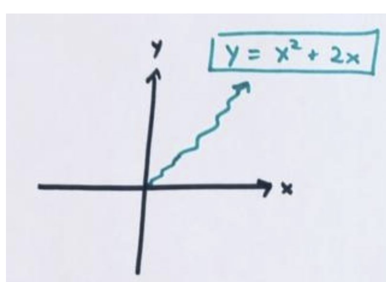
For $x=2$*, we'll add* $(2)^2$ *and* $2(2)$ *and get* 8.

For $x=10$ *we'll add* $(10)^2$ *and* $2(10)$ *and get* 120.

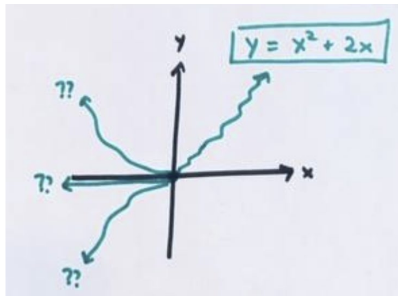
That that is, we are plotting the graph of the equation $y=x^2+2x$.



It is clear as that as we put in larger and larger positive x *values, we are getting higher and higher points. The graph rises upwards as we move to the right. (Maybe in a straight line? Maybe in some curved way?)*



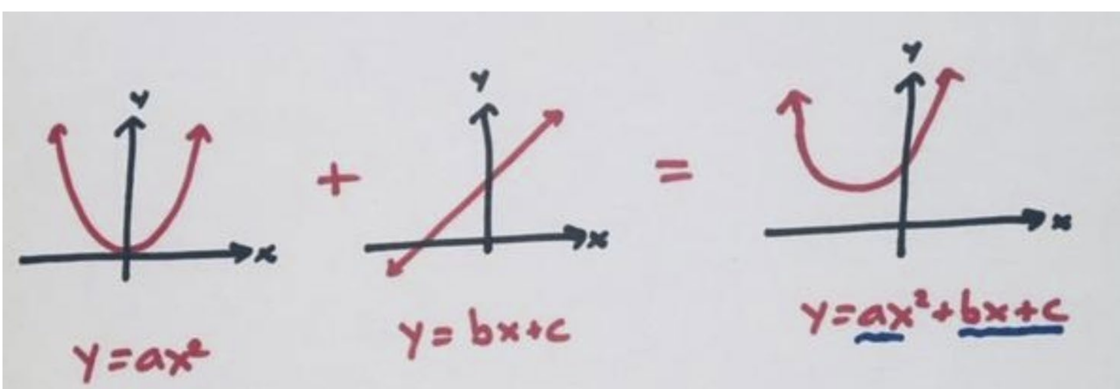
Things are more interesting to the left for negative x values: the graph of $y = x^2$ has data points with positive heights in this region, but the graph of $y = 2x$ has points of negative heights here. We'll be adding together positive and negative heights. Will they cancel out and give zero heights? Will the positives heights "beat" the negative ones and give an overall graph of positive height? Or will the negative heights "win"?



Do plot the graph of $y = x^2 + 2x$. Make a table of data values that give true number sentences and plot those data points. Consider enough of them to get a good sense of the shape of the graph. You are in for a surprise!

The surprise is that it is another perfectly symmetrical U-shaped graph just positioned at a different place in the plane. STUNNING and SHOCKING!

In fact, adding together any basic quadratic expression $y = ax^2$ and a linear expression $y = bx + c$ is sure to give an expression whose graph is the same symmetrical U-shaped graph, maybe flattened or steepened a bit, and maybe shifted to a new position in the plane.



This is just astounding!

That U-shape is astonishingly robust and one cannot help but be transfixed by its robustness.

Of course, one can explain why this graphing phenomenon holds by analyzing the algebra of quadratic equations. But hopefully you were equally shocked by this phenomenon when you first started learning to graph quadratics.

Comment: If you are looking for a review of quadratics, their algebra and their graphs, full of all the surprises and shocks, see www.gdaymath.com/courses.

Now to the issue we'll explore in these notes.

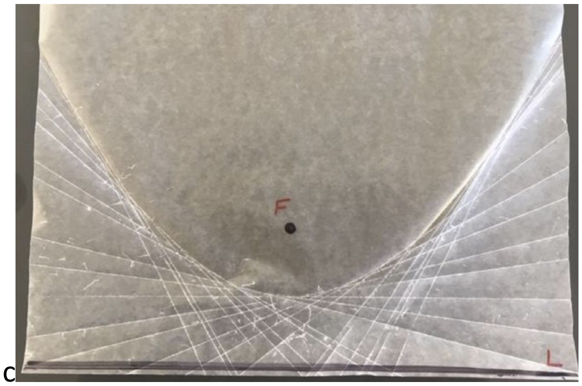
How ubiquitous are these U-shaped curves from quadratics?

Are the U-shaped curves we see in all sorts of contexts basically the same U-shape, (maybe, as I said, just steepened or flattened a bit and moved or even turned upside-down perhaps)? Are these curves so robust so as to be universal in nature?

For example, school physics tells us that the path of a thrown object follows the arc of a quadratic graph. Is that true? How do we know?

Famous Italian mathematician and physicist Galileo Galilei (1564-1642) wondered about the shape a chain hanging between two poles makes. (We see this shape in hanging power lines, in the shape of ropes that surround sculptures in art museums, and so on.) Is this the same quadratic U-shape? How could we know?

Greek scholars of ancient times used geometry to describe all sorts of special curve. They called one of their curves a *parabola* and it too is U-shaped.



c

Algebra wasn't invented for another 800 years or so and so these scholars did not ask: *Is this curve given by a quadratic equation?* But we can! So, is it? How could we know?

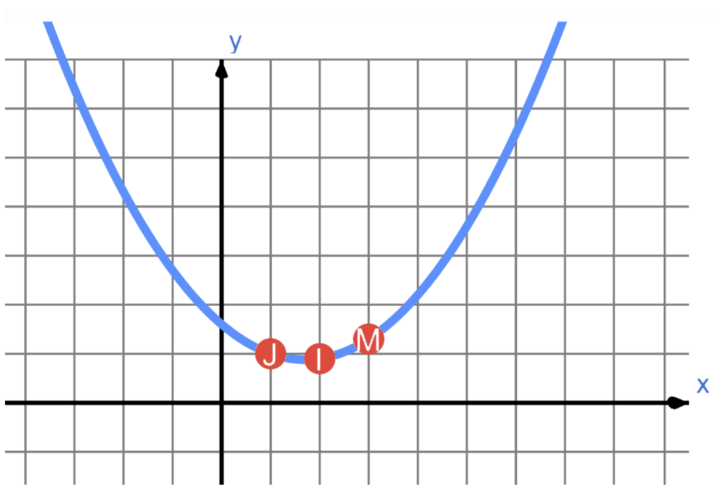
Is the famous St. Louis Gateway Arch in Missouri of the United States in the shape of a quadratic curve?

Is the graph of the equation that spells my nickname, JIM, quadratic? This curve passes through the points

$x=1$, $y=10$ and the 1st letter of my name is the 10th letter of the alphabet, J;

$x=2$, $y=9$ and the 2nd letter of my name is the 9th letter of the alphabet, I;

$x=3$, $y=13$ and the 3rd letter of my name is the 13th letter of the alphabet, M.



Let's answer all these questions—and more! We're in for a cool adventure!

Lesson 2: GRAVITY and GALILEO'S INSIGHT

Many scholars of ancient times, including Aristotle (ca. 350 BCE), believed that heavier objects fall through the air with faster acceleration than lighter objects. It is hard to detect whether or not this is true with real-life experimentation.

> *Try dropping a lacrosse ball and a tennis ball (they are close to the same size) simultaneously from the same height. Can you tell if one hits the ground before the other?*

Nonetheless, it does seem reasonable to suspect that gravity has a "greater effect" on heavier objects causing them to speed up faster than lighter objects when they fall. Many scholars reasoned this.

But Galileo from the 1500s questioned this reasoning.

He said to imagine two objects of the same size and shape, but of different masses, being dropped from the Leaning Tower of Pisa at the same time. If gravity has stronger effect on the heavier object, then it would land first, taking the shorter amount of time to reach the ground.

Now attach a very light string between the two objects, ostensibly making them one object which is of greater mass than either individual object. Gravity then should have an even stronger effect on this system. So the two linked objects dropped from the Tower should fall to the ground in shorter time still. But how do the two objects now "know" they are attached as one object and should fall faster? This doesn't make sense!

Galileo concluded then that all objects of the same size and shape must fall through the air in unison, irrespective of their masses. (Air resistance causes objects to fall at different rates according to their shapes: a sheet of paper is slower to fall than a paper-clip of the same mass, for instance. But it is not gravity causing this variation.)

Letters and biographies about Galileo written at and near the time say that he considered going to the top of the Leaning Tower of Pisa to drop and time falling objects so as to verify that their accelerations to the ground were the same in value. But there is no evidence to suggest that he actually did this. (But he did mimic the experiment by rolling objects down ramps. Their slower motion made is possible to time their rates of descent due to gravity.)

> **Aside:** In 1971, Apollo 15 Commander David Scott while walking the surface of the Moon performed a live television demonstration in which he simultaneous dropped a hammer and a feather to see if, in the absence of an atmosphere, they would indeed accelerate to the ground at identical rates. You can see the video here.
> https://www.youtube.com/watch?v=5C5_dOEyAfk

What Galileo was Expecting to See

Let's imagine Galileo did drop objects from the top of the tower and could time their falls. What was he expecting to see from his data?

Here's a completely made-up data set that assumes the tower is 100 feet tall and that an object took 2.5 seconds to fall to the ground. (Is this even close to being realistic?)

t	0.0	0.5	1.0	1.5	2.0	2.5
h	100	96	84	64	36	0

Here t represents the time passed (measured in seconds) since dropping the object and h the height of the object at each time (measured in feet).

What should data like this reveal about acceleration if Galileo's reasoning is correct?

Well,

> *velocity* is the rate of change of position (in our case, heights)
>
> *acceleration* is the rate of change of velocity.

Since the data is based on regular time differences, the velocity of the falling object can be studied by taking the differences in height values. That is, the first row of a difference table for the height data informs us about velocities.

	100		96		84		64		36		0
Velocity →		−4		−12		−20		−28		−36	

And information about acceleration can be gleaned by looking at the changes of velocity, the second row of the difference table.

	100		96		84		64		36		0
		−4		−12		−20		−28		−36	
Acceleration →			−8		−8		−8		−8		

Galileo was hoping to see constant second differences in his data.

Galileo's Results

In his experiments Galileo did indeed verify that acceleration due to gravity seems to be constant for all falling (and ramp-rolling) objects and is independent of the mass of the individual objects. Data from falling motion (ideally) gives constant second differences, provided the initial time measurements are taken at regular intervals.

Galileo then said that it follows that the change in height of a falling object is given by a quadratic expression,

$$at^2 + bt + c.$$

Whoa! Hold on! How does that follow?

This matches what we are told in physics class, but why does it follow? What is it about sequences that allows us to immediately know that the formula behind the data is quadratic?

It seems we need to understand the mathematics of sequences of numbers that have some structure within the differences between terms.

Lesson 3: Sequences – For when you Trust Patterns. (But Please Don't Trust Patterns.)

Let's do a "deep dive" into special sequences of numbers.

If you do believe that patterns likely hold true, what then would you say is the next number in this sequence?

No doubt you noticed the constant difference of 3 between the terms we see and so would guess the next entry to be 20. Of course there is no reason to believe that all differences will forever remain constant, but noticing this for what we have makes 20 an intelligent guess for the next number.

Of course not all sequences have constant differences. For example, this sequence fails to have a constant difference between terms.

But it does have constant "second differences."

A sequence like this one

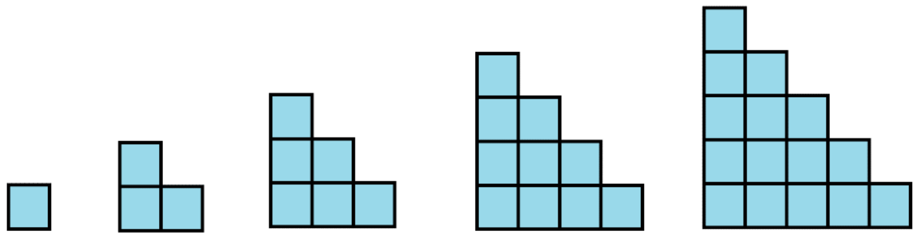
17 17 17 17 17 17 17 17 …

is constant from the start.

Practice 1: *Make an intelligent guess as to the next number in the sequence*

2 3 6 11 18 27 38 __.

Practice 2: *Consider the following sequence of diagrams each made of squares* 1 *unit wide.*



If the implied geometric pattern from these first five figures continues …

a) *What would the perimeter of the tenth figure likely be?*
b) *What would the area of the tenth figure likely be?*

Practice 3:

a) *Show that for the following sequence it seems that the third differences are constant. Make a prediction for the next number in the sequence.*

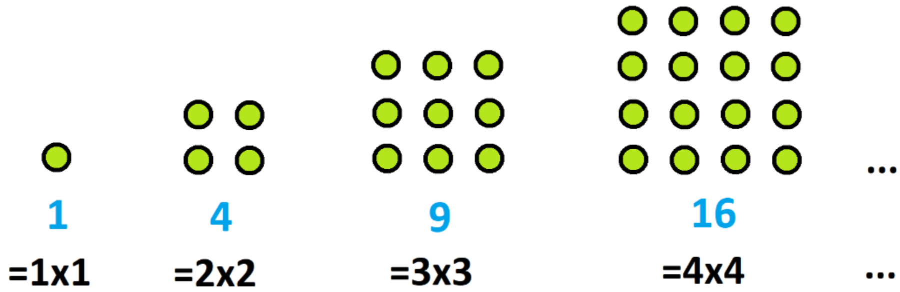
0 2 20 72 176 350 612 …

b) *How many differences must one complete in the sequence below to see a row of constant differences? (The sequence is the powers of two.)*

1 2 4 8 16 32 64 128 256 …

Practice 4:

a) *The sequence of square numbers begins* **1**, **4**, **9**, **16**, **25**, **36**, **49**, **64**, …. (The nth number in this sequence is n^2.)



Is there a row in the difference table of the square numbers that is constant?

a) *The sequence of cube numbers begins* **1**, **8**, **27**, **64**, **125**, **216**, **343**, **256**, …. (The nth number in this sequence is n^3.) *Is there a row in the difference table of the cube numbers that is constant?*

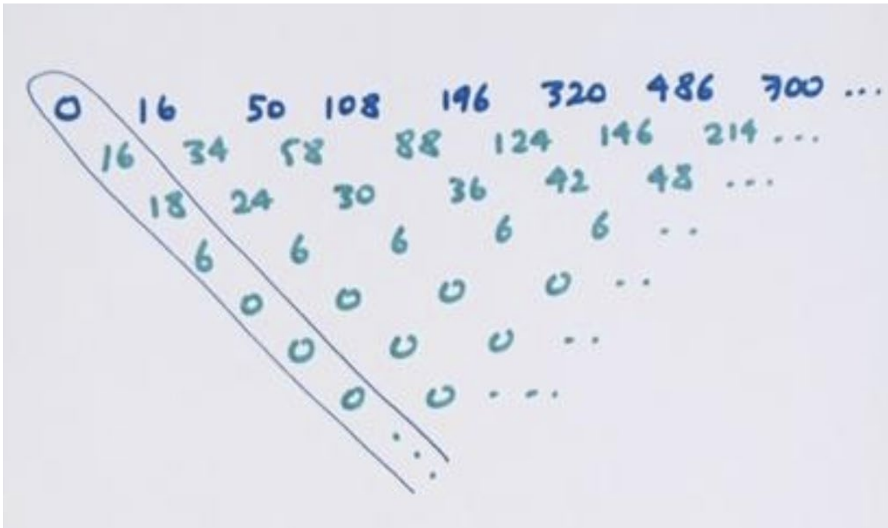
Practice 5: *Use differences to make an intelligent guess as to the next element of this sequence.*

-1 4 7 8 7 4 -1 __

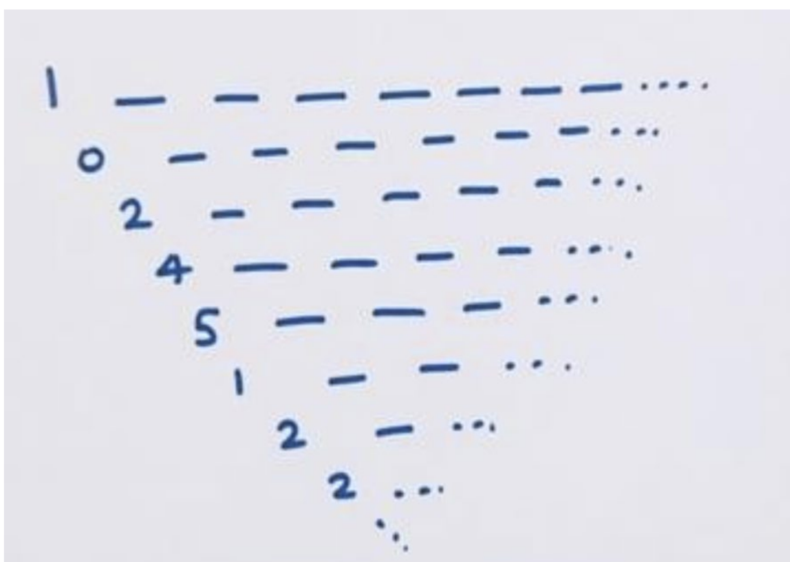
LEADING DIAGONALS

Here is a sequence and a table of all its differences! (Well, if we trust patterns, it is clear that the difference rows are eventually all zero.)

The first entry in each difference row form the *leading diagonal* of the difference table.



And here now is just a leading diagonal of some difference table. Check that you can construct the original sequence from it. For example, the second entry on the top row must differ from 1 by zero, and so also be 1. The second entry on the second row must differ from 0 by two, and so be 2. This means that the third entry on the top row must be 3. (Do you see why?) And so on.



Practice 6: *Do this! Show that you can fill in the entire set of blanks in a difference table just from knowing the table's leading diagonal.*

Practice 7: *What sequence has* **0 0 1 0 0 0 0 0 0 …** *as its leading diagonal?*

0 _ _ _ _ _ _ _ …
0 _ _ _ _ _ _ …
1 _ _ _ _ _ …
0 _ _ _ _ …
0 _ _ _ …
0 _ _ …
⋱

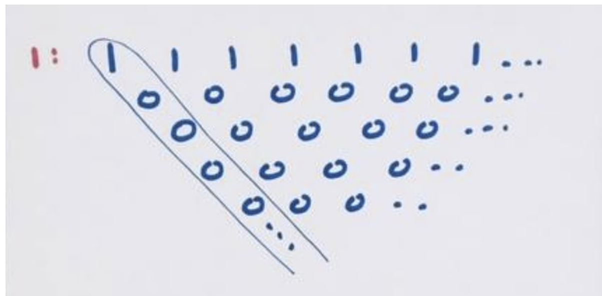
We learn

To know the leading diagonal of a difference table is to know the original sequence!

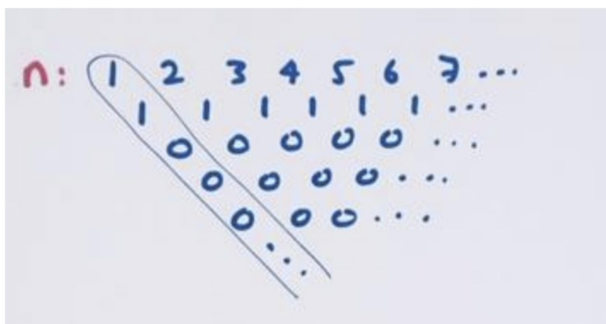
So let's now get to know some leading diagonals of sequences. We'll make that the start of our next lesson.

Getting Formulas from Leading Diagonals

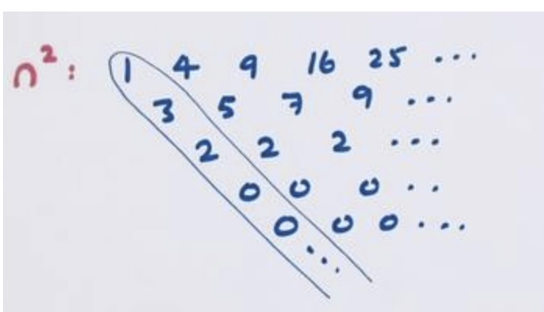
Here's the constant sequence of 1s and its leading diagonal. The nth term of this sequence is, well, 1!

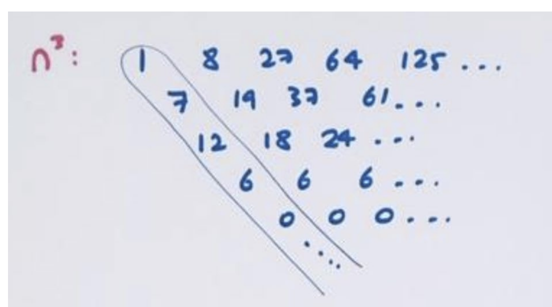


And here's the sequence of counting numbers and its leading diagonal. The nth term of this sequence is n.



A previous practice question looked at the sequences of square numbers (with nth term given by n^2) and cube numbers (with nth term given by n^3). Here are their leading diagonals.





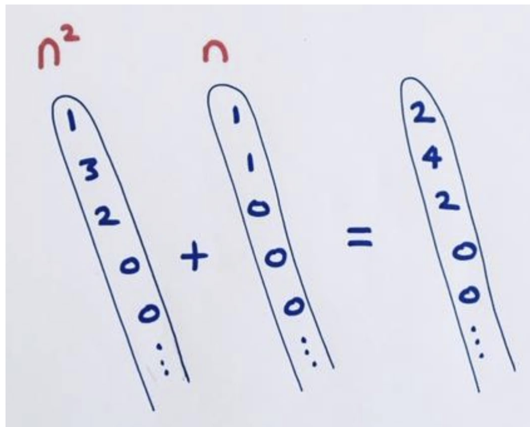
Optional: Care to work out the leading diagonal for the sequence of fourth powers too? (Here the nth term of the sequence is given by n^4.) The answer could be no!

Now consider this.

Look at the sequence with nth term given by $n^2 + n$.

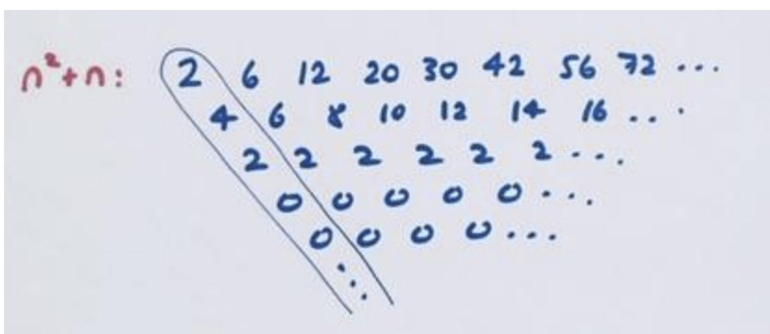
The first term of the sequence is $1^2 + 1 = 2$, the second term is $2^2 + 2 = 6$, the third term is $3^2 + 3 = 12$, and so on. We get sequence **2**, **6**, **12**, **20**, **30**, **42**, ….

Is the leading diagonal of this sequence just the sum of the diagonals for the sequences given by n^2 and n (adding matching entries in each diagonal, perhaps)?



Let's check.

Here's the difference table for the sequence given by $n^2 + n$.



Yes! Its leading diagonal is indeed the sum of the diagonals for n^2 and for n. Wow!

Comment: I am getting a bit loose with my language. Instead of saying "the leading diagonal for the sequence with nth term given by n^2," for instance, I am just saying "the leading diagonal for n^2." I hope my short-hand language is clear enough.

This example suggests a strategy for finding formulas for sequences from our catalogue of standard leading diagonals.

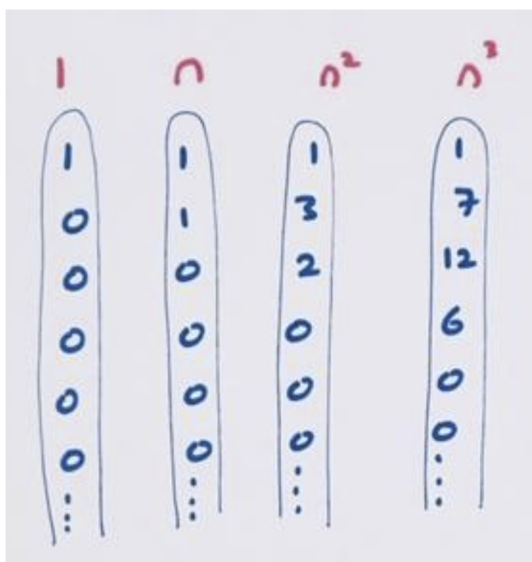
When given a sequence of integers

1. **Find its leading diagonal by constructing its difference table.**
2. **Try to recognize that leading diagonal as a combination of standard leading diagonals. This will give a candidate formula for the sequence.**

This is, of course, assuming we trust patterns and choose to believe that everything just hangs together beautifully and magically. But since we are only dealing with a finite number of terms of any sequence, we can always check our potential formula by plugging in values for n and seeing if does indeed produce that desired list of numbers. We need a third step.

3. **Check to see if the candidate formula actually works.**

For the record, here again are our standard leading diagonals.



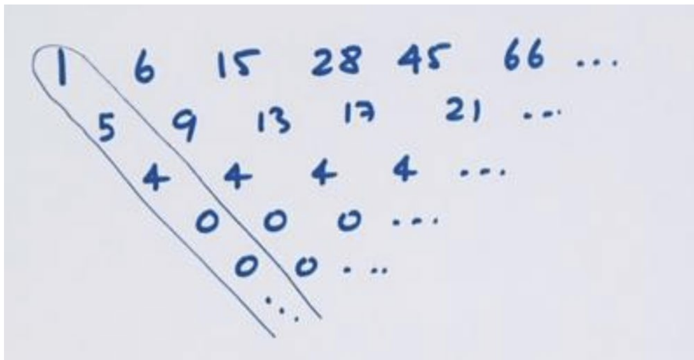
(Feel free to add to this picture the diagonals for the fourth, fifth, sixth, … powers too!)

Example: *Find a formula for the sequence*

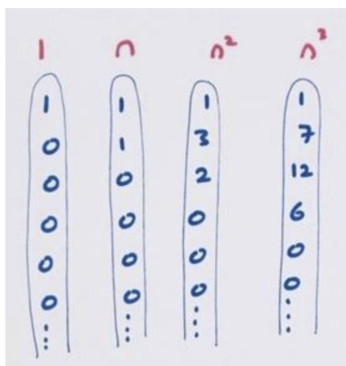
1 6 15 28 45 66 …

(This is, of course, under the assumption we can trust patterns.)

Answer: Here is the difference table for the sequence and its leading diagonal (assuming we can trust the pattern of zeros continues).



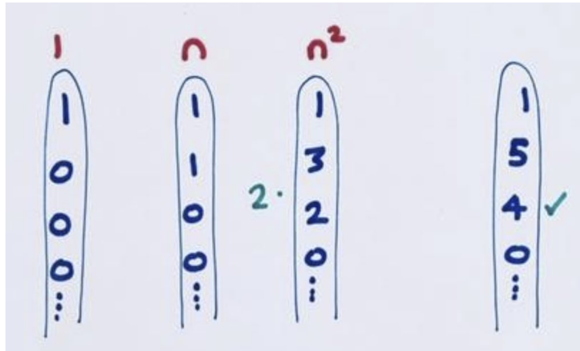
Is this leading diagonal some combination of our standard diagonals?



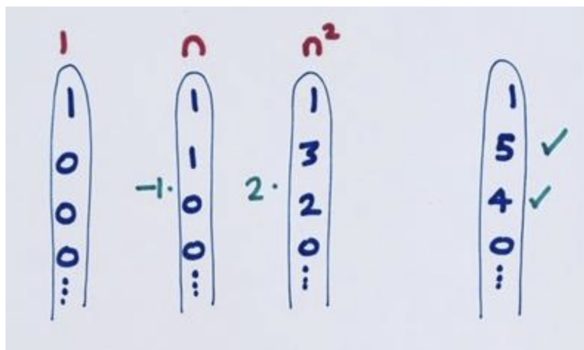
Notice that our leading diagonal has 0s in the fourth position onwards. This tells us we likely won't need to use the diagonal for n^3 (or for any higher of n if you tried working out the diagonals for the fourth powers and the like) as it has non-zero terms in those positions.

So let's focus on using the diagonals of 1 and n and n^2.

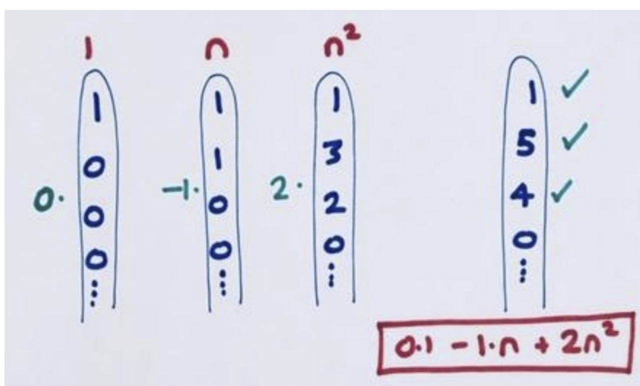
Our leading diagonal has the number 4 in the third position and only the diagonal n^2 has a non-zero entry in the third position. It looks like we'll need two copies of n^2.



We need 5 in the second position and right now we have double 3, to give 6, in that position. This shows we need -1 copies of the n diagonal.



The 4 and the 5 in the target diagonal are now all set, and so is the beginning 1 by luck. We don't need the 1 diagonal at all.



So our candidate formula for the sequence is $2n^2 - n$. And putting in 1, 2, 3, 4, 5, and 6 for n in turn confirms this formula gives the outputs 1, 6, 15, 28, 45, and 66.

We have found a formula for the sequence!

Practice 8: *Use difference methods to find a formula for the sequence of numbers*

2, **2**, **4**, **8**, **14**, **22**, **32**, …

(Just so you know, the answer is $n^2 - 3n + 4$. Can you see get this from looking at the leading diagonal for the sequence?)

Practice 9: *Find a formula that fits the sequence* **0**, **2**, **10**, **30**, **68**, **130**, **222**, ….
(The answer is $n^3 - 3n^2 + 4n - 1$.)

Practice 10:

a) *Find a formula that fits the sequence* **5**, **8**, **11**, **14**, **17**, **20**, **23**, ….
b) *Find a formula that fits the sequence* **3**, **3**, **3**, **3**, **3**, **3**, **3**, ….
c) *Find a formula that fits the sequence* **1**, **3**, **15**, **43**, **93**, **171**, **283**, ….
d) *Find a formula that fits the sequence* **1**, **0**, **1**, **10**, **33**, **76**, **145**, **246**, **385**, ….

e) *Find formulae for as many of these sequences you feel like doing!*

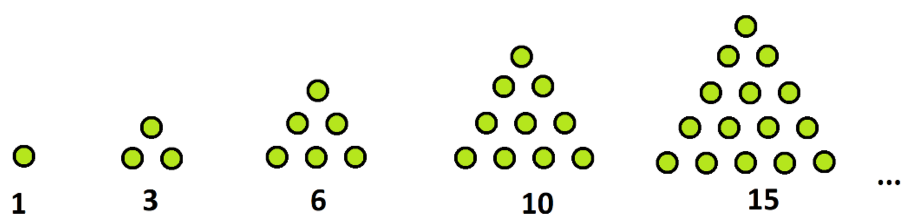
3 3 7 21 51 103 183 297 …

0 9 24 45 72 105 144 189 240 …

6 24 60 120 210 336 …

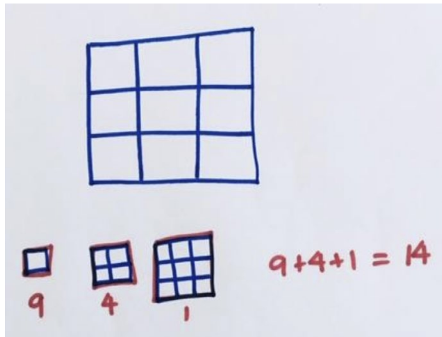
230 275 324 377 434 495 …

Practice 11: *Find a general formula for the* n *th triangular number:* **1**, **3**, **6**, **10**, **15**, **21**, **28**, **36**, …



(Don't be afraid of fractions!)

Practice 12: *Let $S(n)$ be the total number of squares, of any size, one can find in an $n \times n$ grid of squares. For example, $S(3)=14$ because one can find nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of $9+4+1=14$ squares in a 3×3 grid.*

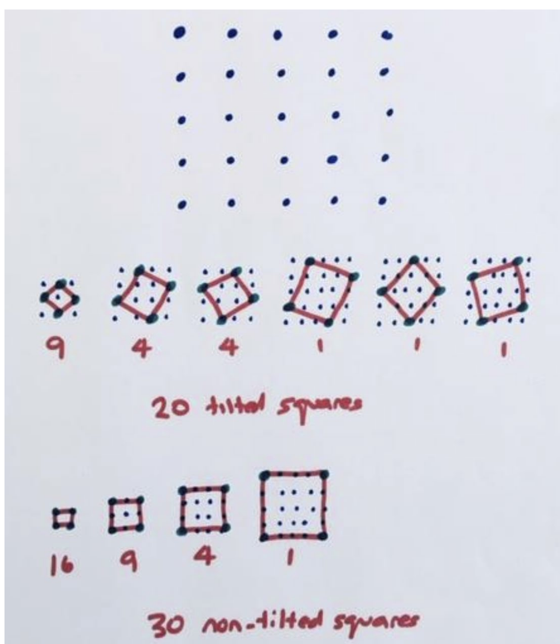


a) *Find $S(1)$, $S(2)$, $S(4)$, and $S(5)$.*

b) *What do our general difference methods suggest for a general formula for $S(n)$?*

c) **OPTIONAL CHALLENGE:** *What is the value of $1^2+2^2+3^2+\cdots+99^2+100^2$?*

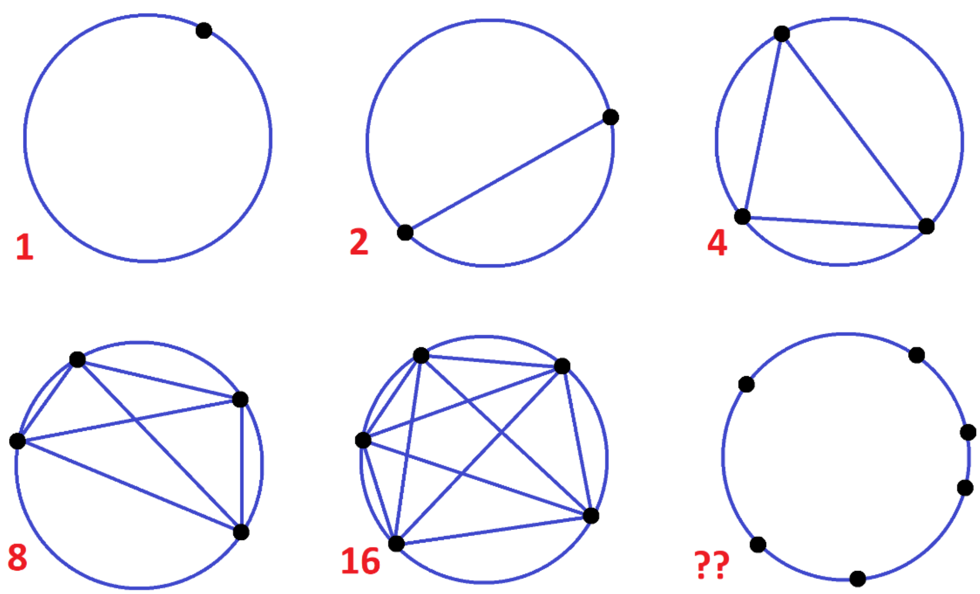
d) **OPTIONAL CHALLENGE:** *Care to count titled and non-tilted squares on arrays?*
For instance, on a five-by-five array of dots on can draw 30 non-tilted squares and 20 titled squares giving 50 squares in total.



But We Cannot Trust Patterns!

Here’s one of my favourite puzzles.

> Draw some circles with dots on their boundaries: one with 1 dot on its boundary, one with 2 dots on its boundary, one with 3 dots on its boundary, and so on.
>
> Next, for each circle, connect each and every pair of boundary dots with a line segment. This divides each circle into a number of regions. Count the regions.

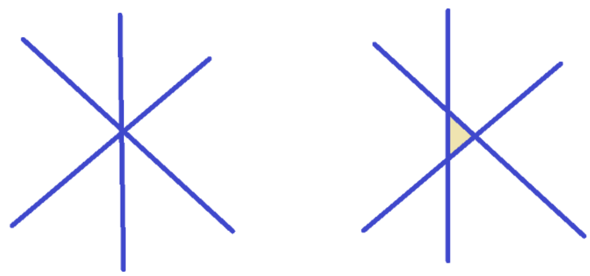


> The circle with one dot has **1** region. The circle with two dots, **2** regions. The circle with three dots, **4** regions. The circle with four dots, **8** regions. The circle with five dots, **16** regions.
>
> How many pieces do you expect to see from six boundary dots?

Up to now we’ve been trusting patterns, and our trust clearly says to expect **32** regions to appear from connecting six dots. (The count seems to double every time.)

But try as you might, you will not get 32 pieces!

Try it! You will either see 30 or 31 regions depending on how you place your dots. (If you are a person who likes to space one's dots symmetrically about the boundary, for instance, you will see only 30 pieces. You will create a multiple intersection point that masks an extra region.)



So here is the moral of this puzzle.

We cannot trust patterns!

We can be excited by patterns. We can be motivated by patterns. But we can never trust a pattern until we have a logical reason to justify why it should be true.

People sometimes say that mathematics is about finding patterns. That is not quite right. It is about finding patterns and structure AND THEN using logical reasoning to explain why those patterns are true and that structure holds—or to find counter examples to show that one's suspicions are wrong.

CHALLENGE: *Suppose 7 dots are placed on the boundary of a circle in such a manner that the maximal number of regions result when one connects pairs of dots with line segments. How many regions is that?*

What is the maximal number of regions that can result with 8 dots? With 9 dots?

Care to keep computing more and more terms of our sequence?

1 2 4 8 16 31 __ __ __ __ __ …

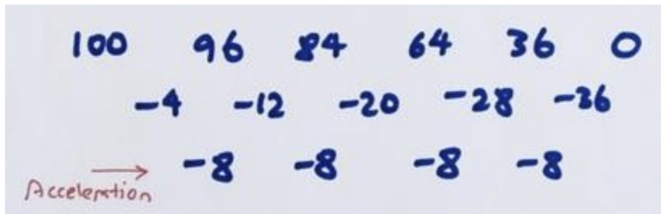
ULTRA-CHALLENGE: *Might there still be a formula for the numbers in this sequence?*

(See chapter 4 of *MATH GALORE!* (MAA, 2012). High-school students found such a formula and proved it works!)

Lesson 4: Projectiles, Parabolas, and Non-Parabolas

Now back to Galileo.

Recall from Lesson 2 that Galileo did confirm experimentally that objects falling (well, rolling) under the influence of gravity accelerate at a constant rate. The data of distances fallen at regular time intervals has constant double differences.



We understand now that Galileo was right to then claim that the data is following a quadratic expression: the height of a falling object is given by a formula of the form

$$at^2 + bt + c$$

where t is the amount of time the object has been falling. Wonderful!

When one throws an object in the air it undergoes two motions:

i) it moves horizontally at a uniform rate (assuming the effect of air resistance is negligible.

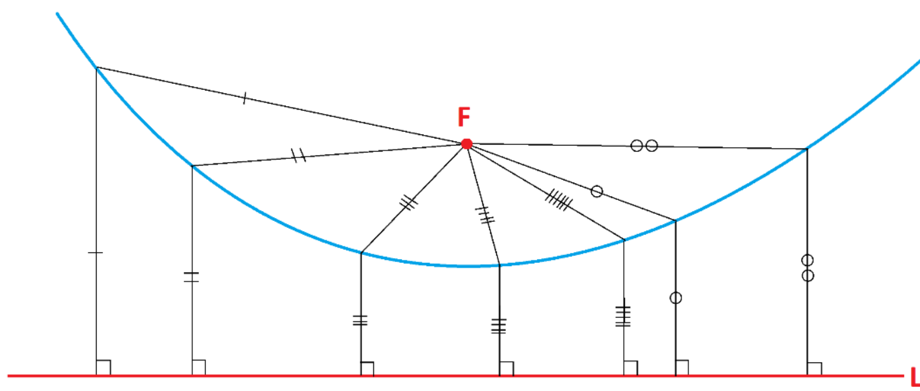
ii) its vertical height changes according to this quadratic formula.

The U-shaped curve one sees as the path of a tossed object is, we have now shown, is indeed the shape of the graph of a quadratic equation.

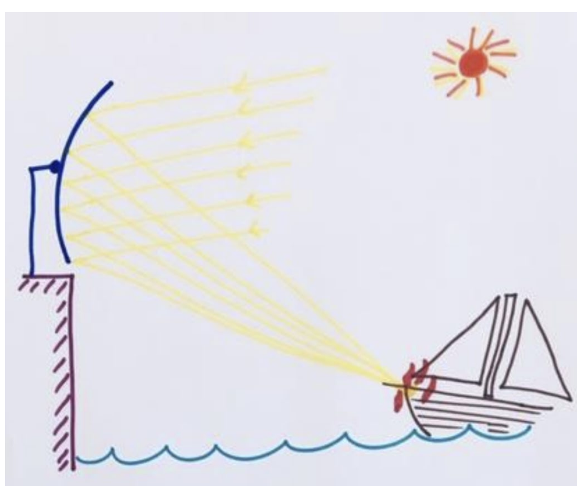
Parabolas

Greeks scholars of ancient times classified a whole host of special curves following certain geometric properties. One such curve is the *parabola*, which we alluded to in Lesson 1. It is constructed as follows.

> Draw a fixed point F (called the *focus*) and a fixed line L (called the *directrix*) on the page. Then the *parabola* with this focus and directrix is the set of all points in the plane equally distant from F and L as shown.

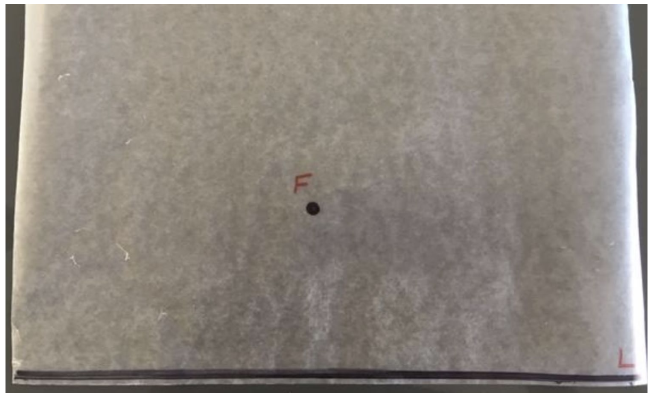


Legend has it that Archimedes of Syracuse (288 – 212 BCE) suggested building huge parabolic mirrors on the cliffs of the island to focus the Sun's rays on approaching enemy wooden-ships and set them on fire. Greek scholars knew that the parabolic shape has the geometric property of focusing parallel rays of light.

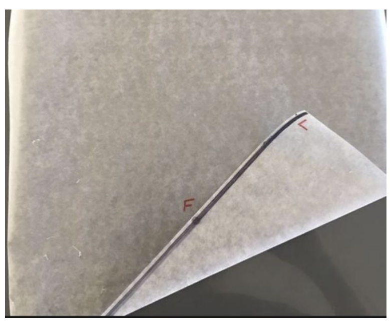


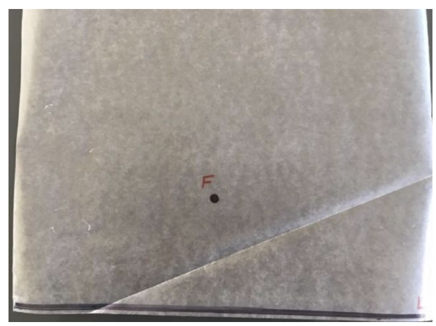
One can construct a parabolic curve by folding paper.

Draw a dot a couple of inches up from the bottom edge of the page for the focus F and imagine the bottom edge line as the directrix L.

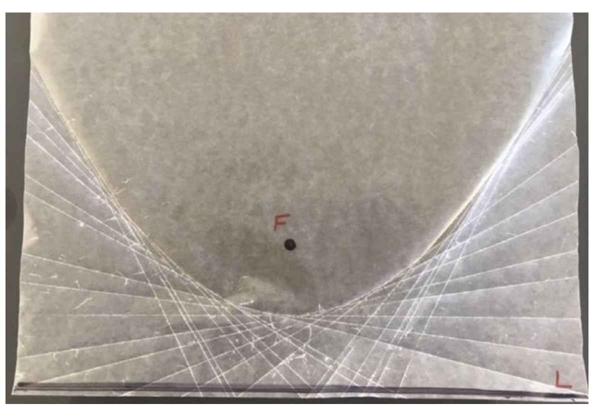


Now lift up the bottom edge and align one point on it with the point F. Make a crease and unfold.





Do this another 50 times or so, lifting different points along the bottom edge up to the point F and making a crease line each and every time.

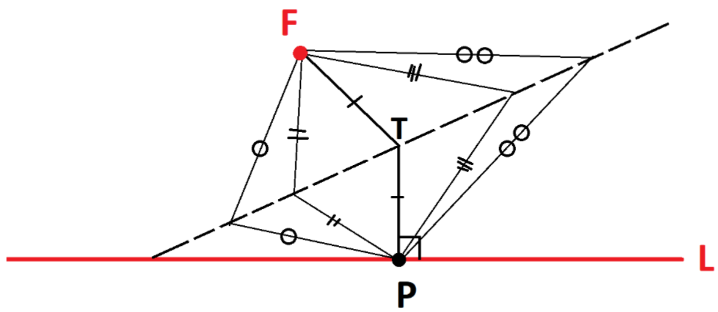


Those crease lines outline a curve, and that curve is a parabola!

ASIDE: Here's how to show that the curve we see has the geometric property of being a parabola.

To be clear, we will show that each crease line we create is a tangent line to a parabola. The set of all crease lines thus give the set of all tangent lines to the parabola.

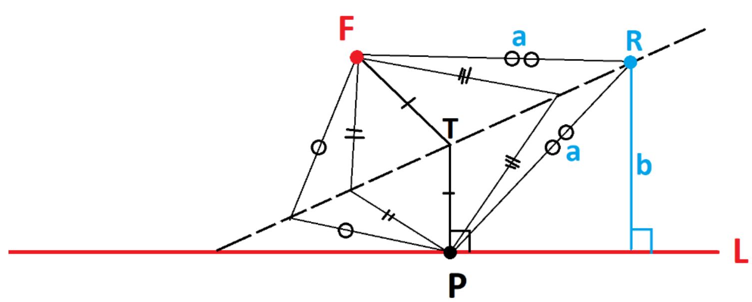
Suppose the point P on the directrix is lifted up onto F to make the dashed crease line shown.



By the symmetry of folding, every point on the crease line is the same distance from F as it is from P. In particular, the point T directly above P on the crease line is equidistant from P and F. We see then that T satisfies the geometric condition of being on the parabola with focus F and directrix L.

Each crease line has at least one point on the parabola with focus F and directrix L.

Consider any other point R on the crease line.



Let a be its distance from F, which equals its distance from P, and let b be its distance from L. We see that $b \neq a$ (b is the leg of a right triangle and a is the hypotenuse and these can't be equal). So R does <u>not</u> satisfy the geometric condition of being on the parabola with focus F and directrix L.

Each crease line has only one point on the parabola with focus F and directrix L.

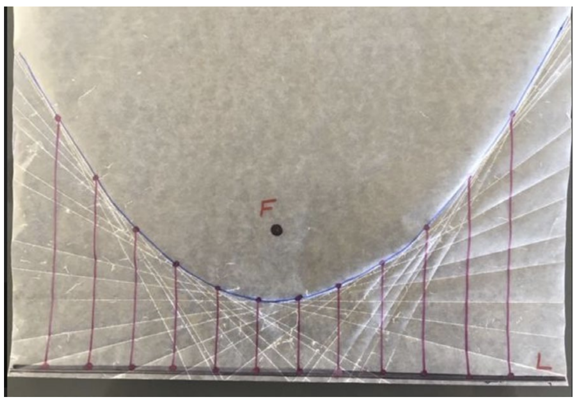
Thus each crease line meets the parabola with focus F and directrix L at only one point, and so is indeed a tangent line to the curve.

ACTIVITY:

High school textbooks make the claim that a parabola is the same shape curve as the graph of a quadratic equation $y = ax^2 + bx + c$. Does this seem feasible?

a) *Make a parabola by folding paper.*

b) *Mark off regular intervals along the bottom edge and measures heights from the bottom edge to the curve as shown. This gives a sequence of values.*



c) *Compute the difference table for this data.*

d) *Within human error, does it seem reasonable to say that you have constant double differences?*

CHALLENGE: *Suppose we situate matters in the coordinate plane so that the directrix L is a horizontal line k units below the x-axis (and so has equation $y = -k$) and the focus F is a point k units high on the vertical axis (and so has coordinates $(0,k)$).*

Translate the geometric condition for a point $P = (x, y)$ to be the parabola with focus F and directrix L into an algebraic condition. Is that algebraic condition a quadratic equation?

HARD CHALENGE: *Conversely, prove the graph of a quadratic equation $y = ax^2 + bx + c$ is sure to be a parabola. What is its focus? What is it is directrix?*

Pedagogical Comment: This is a common faux pas of many high school texts. It is taken as "obvious" that the terms *parabola* and *quadratic graph* are interchangeable. The definition of a parabola is very different from the definition of the graph of an equation $y = ax^2 + bx + c$ and it is not at all obvious that these two curves are, or even should be, the same.

Galileo's Mis-step

Let's keep studying the U-shaped curves we see in nature.

Galileo noticed that the shape of chain hanging between two poles follows a U-shaped curve and did wonder if it is following a basic quadratic formula too. He suspected it did.

ACTIVITY:

a) *Hang a piece of light-weight chain on a white-board.*

b) *With a ruler, draw a horizontal line and mark off regular intervals along it. Measure the horizontal heights shown and collect a sequence of data values.*

c) *Use difference methods to find a quadratic formula for the heights along the chain. Be honest about what you notice.*

DO NOT READ THE NEXT SECTION UNTIL YOU HAVE ACTUALLY TRIED THIS ACTIVITY FOR YOURSELF AND HAVE LOOKED AT YOUR OWN DATA. (Spoilers are coming next!)

There are no doubt inaccuracies in your data due to human error and the first and second differences you calculate are not constant. Go back and try to collect more precise and accurate data.

But you will find the second differences are still not constant.

And try as you might, with more and more accurate measurements, you will fail to see constant second differences.

It turns out that the U-shaped curves of hanging chains simply are not quadratic. And your data is telling you that!

The shape of the curve of a hanging chain is known as a *catenary curve* (from the Latin *catena* for chain). It wasn't until 1691 that mathematicians found a precise formula for this curve—and it is very different from a quadratic expression!

To end this lesson, let me point out that you now have the practical power to test whether or not certain U-shape curves are actually quadratic in structure. Feel free to try these three examples.

EXTENDED ACTIVITY 1: *Is the St. Louis Gateway Arch in the shape of a quadratic curve? Find out by making measurements on a photograph of the arch.*

EXTENDED ACTIVITY 2: *Is a semi-circle given by a quadratic expression? Find out by tracing a pot lid on a piece of paper and making measurements with a ruler.*

EXTENDED ACTIVITY 3: *Is the path of a projectile truly quadratic (or does air resistance have a significant effect on the shape of paths)? Find a time-lapse series of photographs of a basketball or some other tossed object and measure the height of the ball on the photographs at regular time intervals.*

Comment: Here's a shocker. The path of a projectile on the surface of a planet—even one without an atmosphere, which avoids air resistance—is never quadratic in nature! The motion of a thrown object in ideal circumstances is actually the arc of ellipse, another special curve identified by Greek scholars of ancient times. The underlying assumption in physics classes is that the Earth is flat and that gravity always points pulls in the same direction. But on a spherical planet the direction of the "pull" of gravity is always towards the center of the planet, and this direction changes as the object moves.

Lesson 5: Debunking Patterns

We have now seen that not all curves that appear to be quadratic actually are!

Let's see what we can do now if we don't trust the patterns we see in sequences.

Suppose you take an intelligence test and are asked this question.

What is the next number in the sequence?

2, 4, 6, 8, ___

Clearly you would answer 17, because you could see that this sequence is following the formula

$$\frac{7}{24}n^4 - \frac{35}{12}n^3 + \frac{245}{24}n^2 - \frac{151}{12}n + 7.$$

(Test this. Put in $n=1$ and see the formula gives 2. Put in $n=2$ and see it gives 4, put in $n=3$ to see it gives 6, put in $n=4$ to see it gives 8, and finally, put in $n=5$ to see it gives 17.)

Actually, you change your mind, and say the next number is -8 because you recognize the sequence as following the formula

$$-0.75n^4 + 7.5n^3 - 26.25n^2 + 39.5n - 18.$$

(Test this. Put in $n=1$ and see the formula gives 2. Put in $n=2$ and see it gives 4, put in $n=3$ to see it gives 6, put in $n=4$ to see it gives 8, and finally, put in $n=5$ to see it gives -8.)

But then you realize that you can write down any number for the next value, and so you decide to just answer with the symbol A for a general number knowing that

$$\frac{(A-10)}{24}(n-1)(n-2)(n-3)(n-4) + 2n$$

gives the value 2 for $n=1$, 4 for $n=2$, 6 for $n=3$, 8 for $n=4$, and A for $n=5$.

The fact is one can readily create a formula to fit any finite set of data values, including a data set that has a "crazy" final term.

This shows that believing in patterns can be dangerous!

So how does one find formulas to fit data?

We give two methods.

Our first technique for finding formulae for data is due to French mathematician Joseph-Louis Lagrange (1736-1813) and is today called *Lagrange's Interpolation Formula*. It looks shockingly scary at first.

Lagrange's Interpolation Formula

Here is an "obvious" formula that takes the values A, B, C, D, and E for the values $n=1$, 2, 3, 4, and 5 in turn.

$$A\frac{(n-2)(n-3)(n-4)(n-5)}{(-1)(-2)(-3)(-4)}+B\frac{(n-1)(n-3)(n-4)(n-5)}{(1)(-1)(-2)(-3)}$$

$$+C\frac{(n-1)(n-2)(n-4)(n-5)}{(2)(1)(-1)(-2)}+D\frac{(n-1)(n-2)(n-3)(n-5)}{(3)(2)(1)(-1)}$$

$$+E\frac{(n-1)(n-2)(n-3)(n-4)}{(4)(3)(2)(1)}$$

Indeed, this looks horrific at first glance! But it is actually easy to understand after one has taken a deep breath.

Let's process it in stages.

1. First note that there are five terms added together.

2. There is one term for each appearance of the values A, B, C, D, and E.

3. Each term is designed to vanish for all but one of the values $n=1$, 2, 3, 4, and 5.

To make sense of this third statement, put $n=1$, say, into the formula and see what it does to each term of in the expression. For example, the second term is $B\frac{(n-1)(n-3)(n-4)(n-5)}{(1)(-1)(-2)(-3)}$ with the factor $n-1$ in its numerator. When we put $n=1$ into this expression we thus get zero in the numerator of the expression and thus the whole expression equals zero.

The third, fourth, and fifth terms in the sum also have a factor of $n-1$ in their numerators, and so each equal zero when n equals 1.

Only the first term will be non-zero for $n=1$.

Check: *Put $n=4$ into the formula. Which terms vanish for $n=4$? Which one term "survives" for $n=4$?*

4. Each term in the sum is a number times a fraction. The fraction is designed to equal 1 at a specific value of n.

To make sense of this, consider putting in $n=1$ again. The final four terms in the sum are zero and only the first term "survives." But look what happens when we put in $n=1$ into the first term. We get

$$A\frac{(1-2)(1-3)(1-4)(1-5)}{(-1)(-2)(-3)(-4)} = A\times 1 = A.$$

The denominator was designed to match the numerator for this instance and the term has value A. The sum of all five terms equals $A+0+0+0+0=A$ for $n=1$, just as wanted.

Check: *Put $n=4$ into the entire formula and see it gives the value $0+0+0+D\times 1+0=D$, just as hoped.*

Practice 1: *Convince yourself that the formula*

$$8\frac{(n-10)(n-20)}{(-4)(-14)}+122\frac{(n-6)(n-20)}{(4)(-10)}+4600\frac{(n-6)(n-10)}{(14)(10)}$$

gives the value 8 *for* $n=6$*, the value* 122 *for* $n=10$*, and the value* 4600 *for* $n=20$.

Practice 2: *Find a formula that gives the value* 9000 *for* $n=3$*, the value* -45 *for* $n=5$*, and the value* $\frac{2}{3}$ *for* $n=8$*. (Don't bother simplifying your formula.)*

Practice 3: *Write down a formula that fits this data set.*

x	1	2	7	15
y	5	10	8	-3

Practice 4: *You are asked to write the next number in this sequence:*

1 2 3 4 __

You decide to write $\sqrt{\pi}$*. Write down a formula that justifies your answer.*

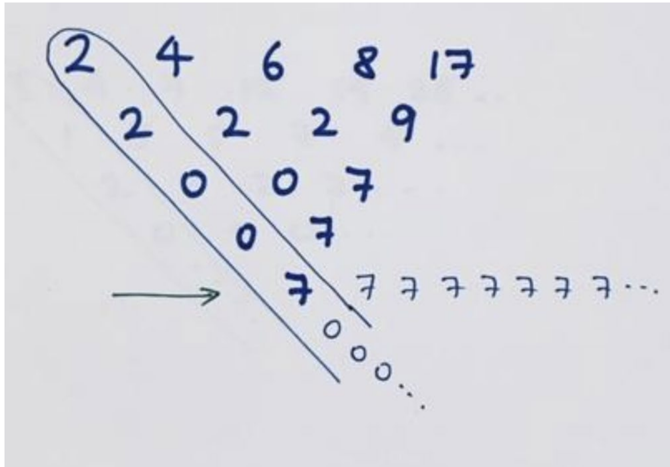
Back to Difference Methods

Here's a second method for finding formulae for finite data sets. It's our earlier approach with difference methods.

Let's go back to the very first sequence of this lesson. Let's find a formula that fits the data

2 4 6 8 17.

To do so, just write out a difference table for those five data values and pretend that the final row is a line of constant differences.



Look at the leading diagonal and recognize it as a combination of standard leading diagonals just as before. If you are not afraid of fractions and working with the leading diagonal of the sequence of n^4 you will get the formula I presented:

$$\frac{7}{24}n^4 - \frac{35}{12}n^3 + \frac{245}{24}n^2 - \frac{163}{12}n + 7 .$$

(Alternatively, use Newton's insight of lesson 8 to get, without any hassle, the same formula, but as $2\cdot 1 + 2\cdot(n-1) + 7\cdot\frac{(n-1)(n-2)(n-3)(n-4)}{24}$.)

Practice 5: *Find an expression* $an^2 + bn + c$ *that fits the sequence* **5 6 10**. *(That is, find a formula that gives* 5 *for* $n=1$*,* 6 *for* $n=2$*, and* 10 *for* $n=3$*.)*

Lesson 6: Fitting Quadratics to Data

Sometimes students are asked to find a quadratic formula $y = ax^2 + bx + c$ to fit three data points. Lagrange's Interpolation formula works very well for this. The task of simplifying the formula one gets (if one is required to simplify answers) is not too onerous.

Practice 1: a) *Show that* $y = 7\frac{(x-4)(x-5)}{(-5)(-6)} + \frac{(x+1)(x-5)}{(5)(-1)} + 10\frac{(x-1)(x-4)}{(4)(1)}$ *is a quadratic equation in disguise. Show that the graph of this equation passes through the data points* $(-1,7)$, $(4,1)$, *and* $(5,10)$.

b) *Find a quadratic formula that fits the data* $(2,5)$, $(-1,6)$, *and* $(5,46)$ *and make your answer look as friendly as possible.*

x	2	-1	5
y	5	6	46

c) *Find a quadratic formula that fits the data* $(2,5)$, $(3,8)$, *and* $(5,14)$ *and make your answer look as friendly as possible. Explain what happens!*

d) *There is no quadratic formula* $y = ax^2 + bx + c$ *that fits the data* $(2,5)$, $(10,6)$, *and* $(10,100)$ *. (The same input value of* $x = 10$ *cannot give two different output values.) So then, how does Lagrange's Interpolation method fail when you try to use it?*

Of course, from the previous lesson we know how to find a polynomial that fits any number of data points, not just three.

Practice 2: *Use Lagrange's Interpolation Formula to find the equation of the line between two points* (p,m) *and* (q,n) *with* $p \neq q$. *Is your equation equivalent to "*$y = mx + b$*" where* m *is the slope of the line segment between the two points?*

ASIDE 7: Personal Polynomials

My proper name is JAMES and I am particularly fond of the polynomial formula

$$p(x) = \frac{83}{24}x^4 - \frac{497}{12}x^3 + \frac{4141}{24}x^2 - \frac{3463}{12}x + 164\,.$$

(We know from the story of *Exploding Dots* that a combination of the non-negative powers of x is called a polynomial.)

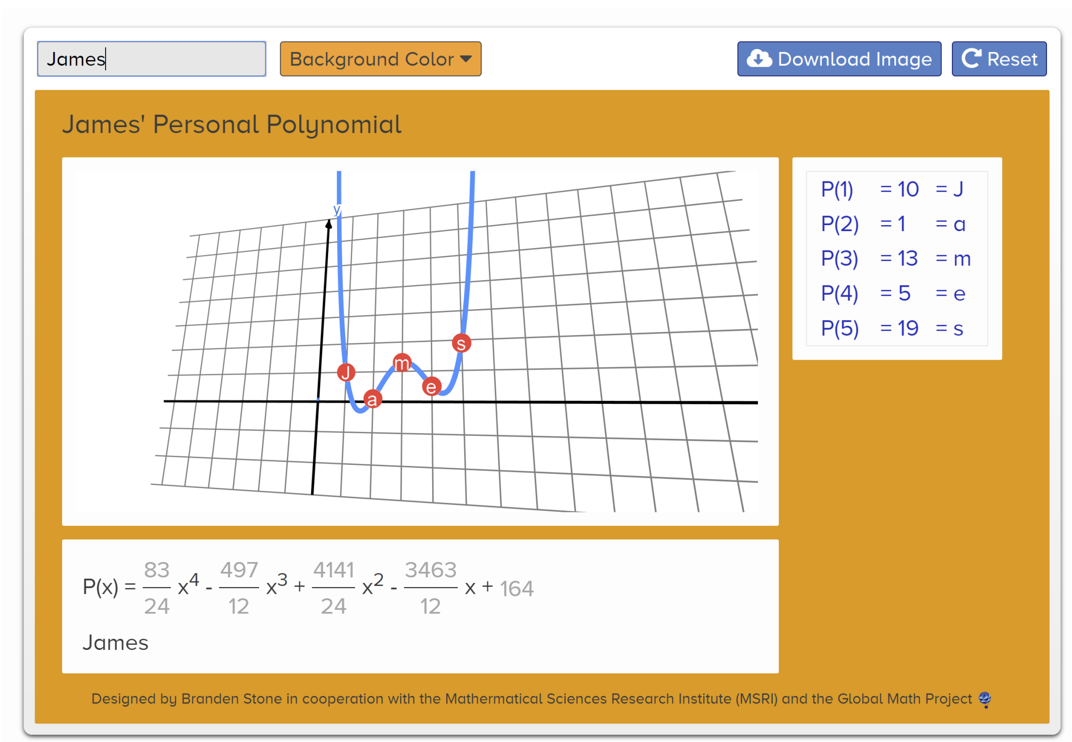
It has

$p(1) = 10$ and the 10th letter of the alphabet is J,
$p(2) = 1$ and the 1st letter of the alphabet is A,
$p(3) = 13$ and the 13th letter of the alphabet is M,
$p(4) = 5$ and the 5th letter of the alphabet is E,
$p(5) = 19$ and the 19th letter of the alphabet is S.

Practice 1: *Find your own personal polynomial!*

Practice 2: *What is the expression that "spells"* JIM*? Is the graph of lesson 1 a quadratic graph?*

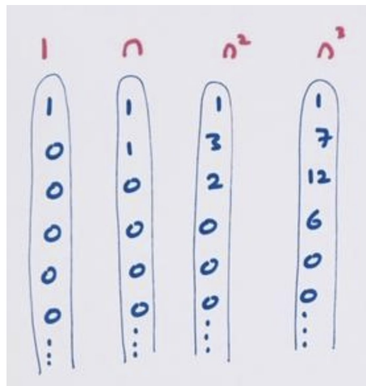
Comment: Go to www.globalmathproject.org/personal-polynomial/ for a really cool web app that finds—and graphs—your personal polynomial for you! (Also see more videos explaining the mathematics.)



OPTIONAL BONUS MATERIAL

Lesson 8: Sequences – For When You Really Do Trust Patterns

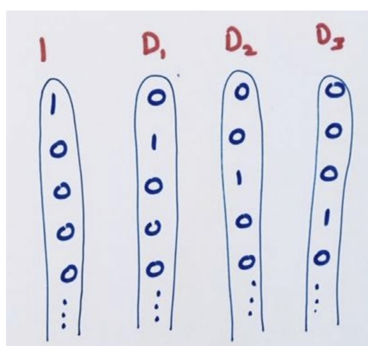
Recall from lesson 3 we were using the standard leading diagonals that come from the sequences generated by the powers of n.



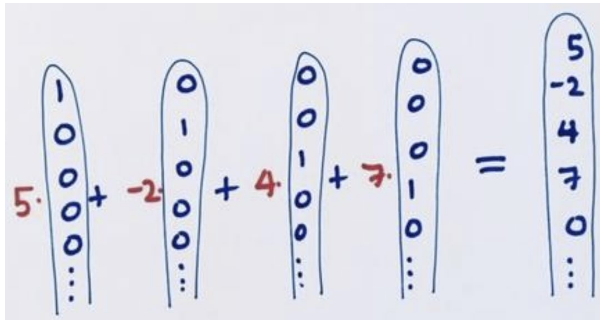
If we trusted seeing rows of constant differences, we could then try to write the leading diagonal of our difference table as a combination of these basic leading diagonals and get a candidate formula for our sequence, which we could then check.

But this set of standard leading diagonals is awkward and hard to work with!

It would be so much easier to work with leading diagonals that look like these.



Then we'd know that the diagonal (5 -2 4 7 0 0 0 …), for instance, is given by $5 \cdot 1 - 2D_1 + 4D_2 + 7D_3$.



We know that the constant sequence of 1s gives us the first leading diagonal. What sequences have D_1, D_2, D_3,and so on, as their leading diagonals?

Exercise: *Fill in each of these difference tables. Do you recognize any of the sequences?*

1 _ _ _ _ _ _ _ …
0 _ _ _ _ _ _ …
0 _ _ _ _ _ …
0 _ _ _ _ …
0 _ _ _ …
0 _ _ …

0 _ _ _ _ _ _ _ …
1 _ _ _ _ _ _ …
0 _ _ _ _ _ …
0 _ _ _ _ …
0 _ _ _ …
0 _ _ …

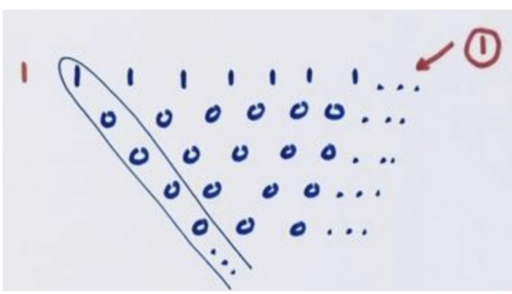
0 _ _ _ _ _ _ _ …
0 _ _ _ _ _ _ …
1 _ _ _ _ _ …
0 _ _ _ _ …
0 _ _ _ …
0 _ _ …

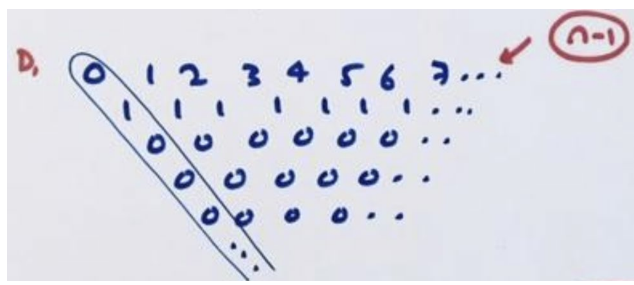
0 _ _ _ _ _ _ _ …
0 _ _ _ _ _ _ …
0 _ _ _ _ _ …
1 _ _ _ _ …
0 _ _ _ …
0 _ _ …

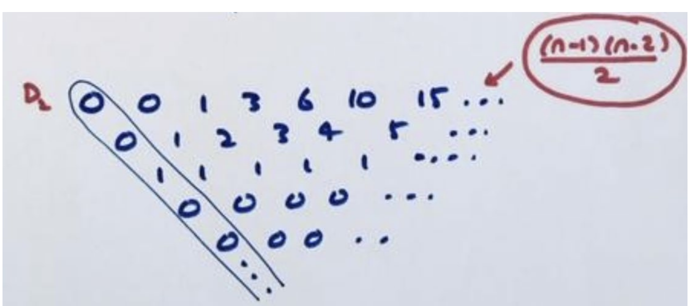
Sir Isaac Newton (1642 - 1727) recognized these sequences as the diagonals of the famous arithmetic triangle, provided one inserts leading zeros into the diagonals.

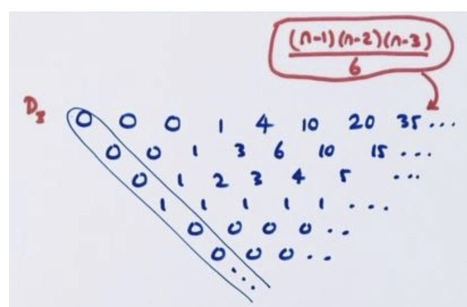
1 0 0 0 0 0
1 1 0 0 0 0
1 2 1 0 0 0
1 3 3 1 0 0
1 4 6 4 1 0
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

And he knew a formula for the terms of this triangle and hence formulas for these terms of each of these sequences. (See part 3 of www.gdaymath.com/courses/permutations-and-combinations/ .)





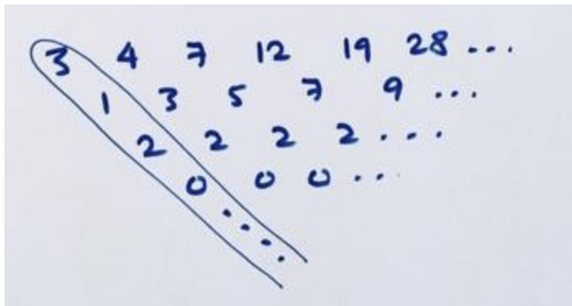




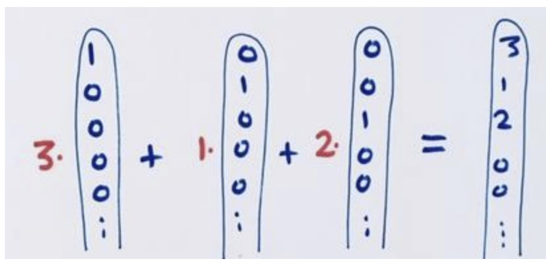
Comment: In general, the sequence that gives the k th diagonal has terms following the formula $\dfrac{(n-1)(n-2)\cdot\cdots\cdot(n-k)}{k!}$.

Example: *Trusting patterns, find a formula for the sequence* **3 4 7 12 19 28 ….**

Answer: We have the following difference table.



And we readily see its leading diagonal as a combination of our (new) standard diagonals.



Thus we suspect this sequence is following the formula

$$3 \cdot 1 + 1 \cdot (n-1) + 2 \cdot \frac{(n-1)(n-2)}{2}$$

which simplifies to $n^2 - 2n + 4$. And one check that this formula does indeed work!

Practice 1: *Trusting patterns, find a formula for the sequence*

3 7 29 99 247 503 897 1459 ….

Practice 2: *Does Newton's approach give the formula* n^2 *for the sequence of square numbers* **1**, **4**, **9**, **16**, **25**, **36**, …? *Does it give the correct formula for the cube numbers?*

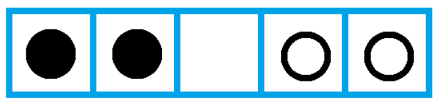
Of course, as we saw, mathematicians never trust patterns. They might be motivated by them, guided by them even, and they might look the formulas they suggest could be true, but mathematicians will never be satisfied until they have a logical explanation that ensures patterns are true.

This next activity illustrates a lovely interplay between play of patterns and logic.

THE SLIDE PUZZLE

This classic puzzle is usually phrased in terms of frogs and toads leap-frogging over each other. Here I'll just use counters.

Let's start with two black and two white counters arranged in a row of five boxes as shown.

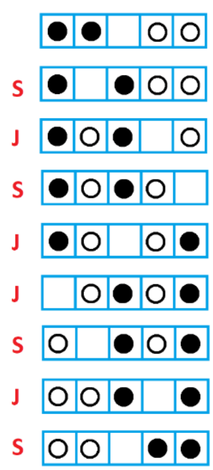


White counters can only move left. Black counters can only move right.

Each move is either a slide (S) in which a counter moves one place over to an empty cell or a jump (J) in which a counter leap-frogs over an adjacent counter (of any color) into an empty cell two places over.

The goal is to have the black and white counters switch positions.

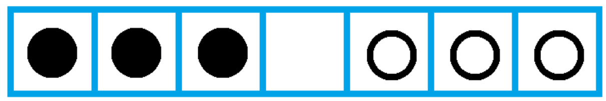
With $N = 2$ counters of each color the puzzle can be solved in 8 moves.



Let $P(N)$ be the number of moves required to solve an analogous puzzle with N black counters and N white counters arranged in a row of $2N+1$ boxes.

We have shown that $P(2)=8$.

a) *Find $P(3)$ by solving the $N=3$ version of the puzzle.*



b) *Find $P(1)$, $P(4)$, and $P(5)$.*

c) *Assuming we can trust patterns, find a possible formula for $P(N)$.*

Solving the $N=2$ puzzle followed the pattern of moves

S J S JJ S J S.

Here we have four slides and four jumps.

d) *What patterns of moves solves the $N=1$, $N=3$, $N=4$, and $N=5$ puzzles? How many slides appear in each? How many jumps?*

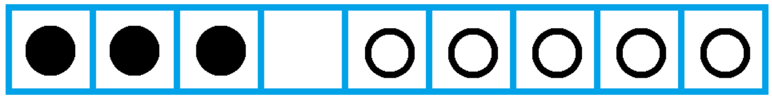
Any conjectures as to how many slides and how many jumps there will be in solving a general puzzle with N black and N white counters?

e) *With N black and N white counters explain why a solution must involve N^2 jumps.* (Use a logical argument this time and don't rely on patterns.)

f) *With N black and N white counters how many places to the left must each white counter move and how many places to the right must each black counter move? Explain why we must have $P(N)=2N(N+1)-N^2$. (Does this match your answer to part c?)*

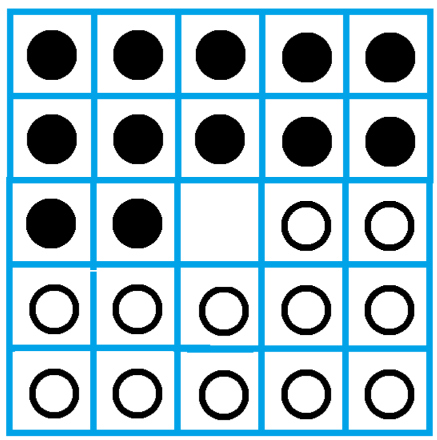
g) *Solve the puzzle again for the case $N=3$. This time watch the location of the blank space. Is each of the seven boxes empty at some point of play? Must this be the case?*

Challenge: Suppose there are M black counters and N white counters.



Is it always possible to rearrange the counters so that the black ones sit to the right and the white ones sit to the left? Is there a pattern to the moves required? Is there a general formula for the number of moves required?

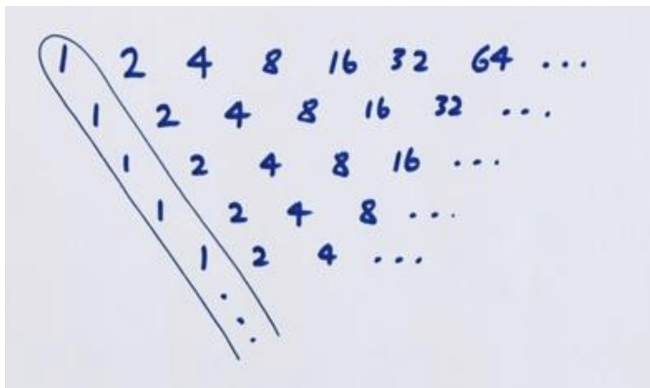
Challenge: Is it possible to interchange the black and white counters in this two-dimensional array? Here counters may move in any direction – left, right, up, down – via slides and jumps.



Differences that Reoccur

Of course, sequences can exhibit many different types of behavior.

For instance, all the sequences we've studied in these notes eventually yield a row of constant values in their difference tables, except one! We saw that the sequence of the powers of two fail to yield a row of constant differences.



So our difference methods will not yield a formula for the powers of two. (But we already happen to that that nth power of two is given by 2^n.)

Notice that the leading diagonal for the powers of two is the constant sequence of 1s. It's actually the sequence of the powers of 1.

a) *Consider the sequence of the powers of three:* **1 3 9 27 81 243 729 ….** *Show that its leading diagonal seems to be the powers of two.*

b) *Consider the sequence of the powers of four:* **1 4 16 64 256 1024 4096 ….** *Show that its leading diagonal seems to be the powers of three.*

c) *Prove that the leading diagonal of the powers of* a

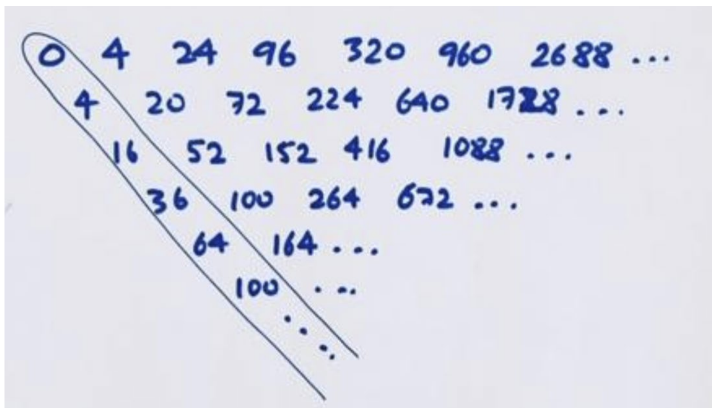
$$1 \quad a \quad a^2 \quad a^3 \quad a^4 \quad a^5 \quad a^6 \quad \cdots$$

is sure to be the powers of $a-1$.

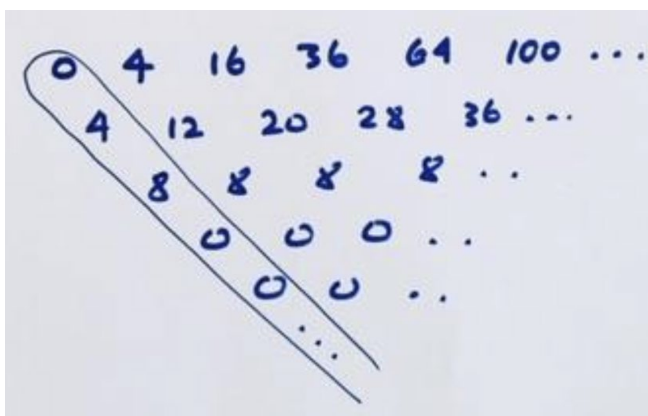
Mathematician Robert Jackson noticed that computing the difference table for a leading diagonal that isn't mostly zeros can lead information about the original sequence.

For example, consider this sequence.

Here's its difference table.

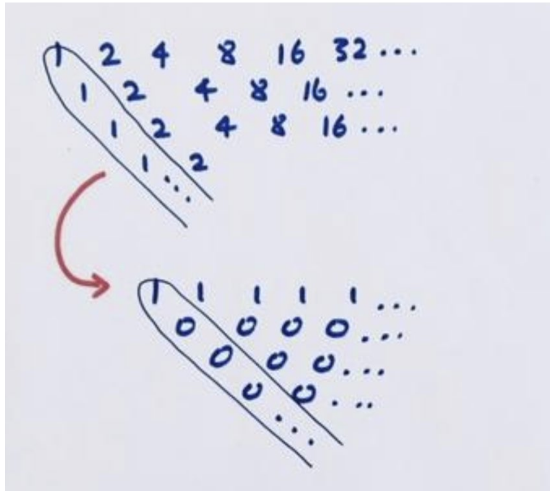


Its leading diagonal doesn't seem to go to zero. So let's compute the difference table of this diagonal.



This now is mostly zero.

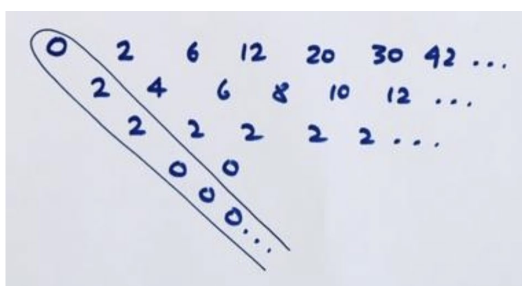
Jackson noticed that this behavior is similar to the behavior of the powers of 2. They too have a leading diagonal that is not going to zero (the powers of 1), but the leading diagonal of that leading diagonal is mostly zero.



So maybe our original sequence is a "contains" the powers of 2? Let's try dividing the terms of sequence by respective powers of 2.

$$\begin{aligned} &0 \div 1 = 0 \\ &4 \div 2 = 2 \\ &24 \div 4 = 6 \\ &96 \div 8 = 12 \\ &320 \div 16 = 20 \\ &960 \div 32 = 30 \\ &2688 \div 64 = 42 \\ &\quad\vdots \end{aligned}$$

Now look at the difference table of this divided sequence.



Our methods give the formula $n^2 + n$ for this divided sequence, suggesting that the nth term of our original sequence must be

$$2^n\left(n^2 + n\right)$$

and we check that it works!

Practice 3: *Consider the sequence of numbers*

0   9   108   891   5832   32805   166212   780759   …

Draw the difference table for this sequence.

Compute the difference table for its leading diagonal.

Compute the difference table for the leading diagonal of this difference table.

Keep doing this for a while and see that the sequence is exhibiting behavior analogous to that of the powers of three.

Find a formula for the n *th term of this sequence.*

Of course, not all sequences succumb to the approaches we described thus far. For example, look at the sequence of Fibonacci numbers with each term after the first two is the sum of its previous two terms.

1 1 2 3 5 8 13 21 34 55 89 …

Do our techniques work to give a general formula for the n th Fibonacci number?

The theory of difference methods is still an area of research for mathematicians.

ASIDE 9: Final - and Optional – Theoretical Comments

In the lessons about sequences I was very careful to point out that we were getting candidate formulas for our sequences and that we had to go to the trouble to check if they actually work. Well, it turns out that they will always work, and now we have the means to prove that they will.

The argument is theoretical, and I'll demonstrate the gist of it here with one specific instance. We'll prove:

> *If a sequence has a difference table with constant second differences (but not constant first differences), then than sequence is given by a formula of the form* an^2+bn+c*, a quadratic formula. Moreover, this is the formula that arises by recognizing the leading diagonal as a combination of the* 1*,* n*, and* n^2 *leading diagonals.*

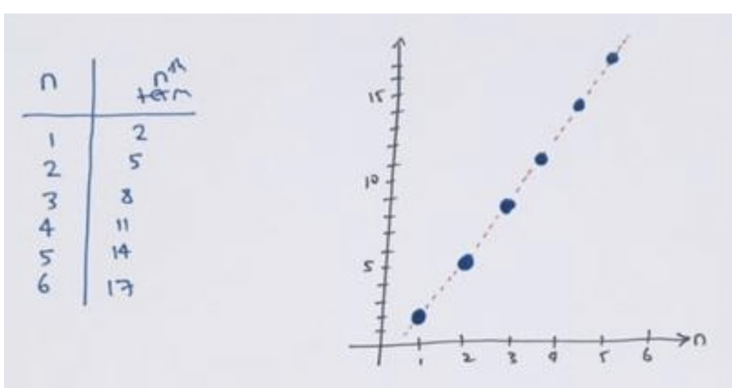
The same approach can be used to prove:

> *If a sequence has a difference with constant third differences (but not constant second differences), then that sequence is given by a cubic formula* an^3+bn^2+cn+d*.*
>
> *If a sequence has a difference with constant fourth differences (but not constant third differences), then that sequence is given by a quartic formula* $an^4+bn^3+cn^2+dn+e$*.*

And so on. We also have

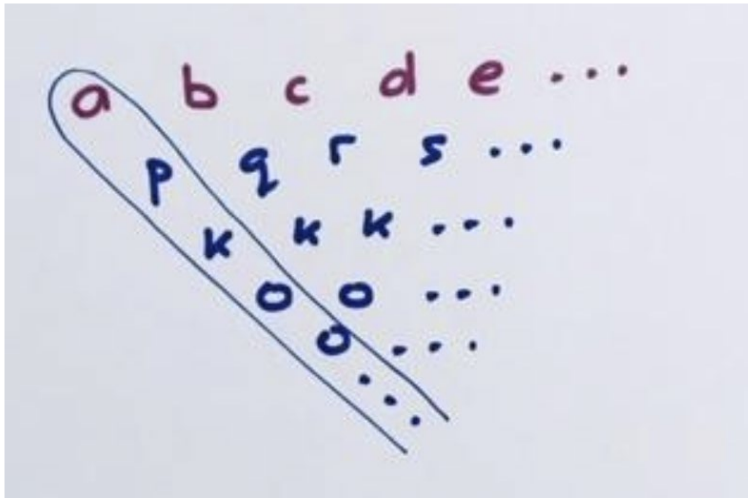
> *If a sequence has a difference with constant differences (but the sequence itself is not constant), then that sequence is given by a linear formula* $an+b$*.*

Our very first example was an example of a "linear sequence". It is generated by the formula $3n-1$.



Let's prove the claims made about a sequence with constant second differences.

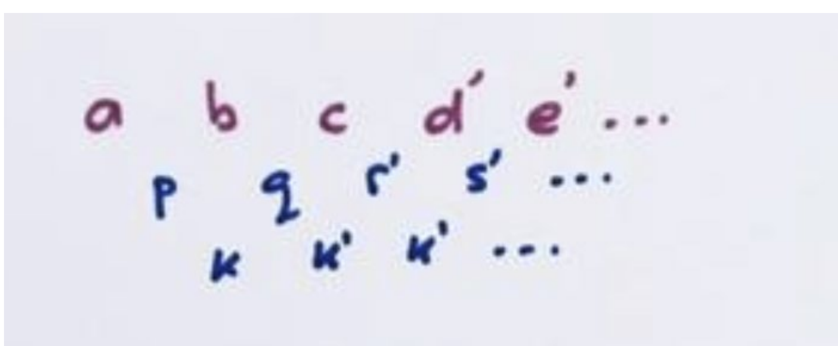
Suppose **a b c d e …** is such a sequence.



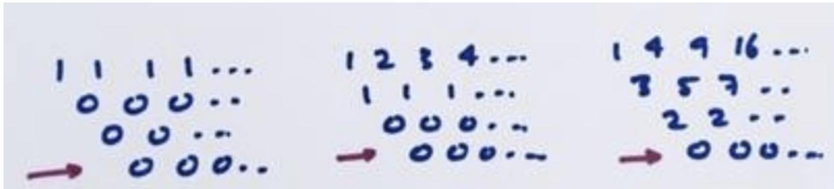
1. Lagrange's interpolation formula shows that there is a quadratic expression that gives the value **a** for $n=1$, **b** for $n=2$, and **c** for $n=3$.

$$a\frac{(n-2)(n-3)}{(-1)(-2)}+b\frac{(n-1)(n-3)}{(1)(-1)}+c\frac{(n-1)(n-2)}{(2)(1)}=An^2+Bn+C$$

2. This quadratic formula generates its own sequence and difference table. Since the first three terms of the first row were designed to match the original sequence, the first two terms of the second row and the first term of the third row of the difference table match the difference table of the original sequence too.

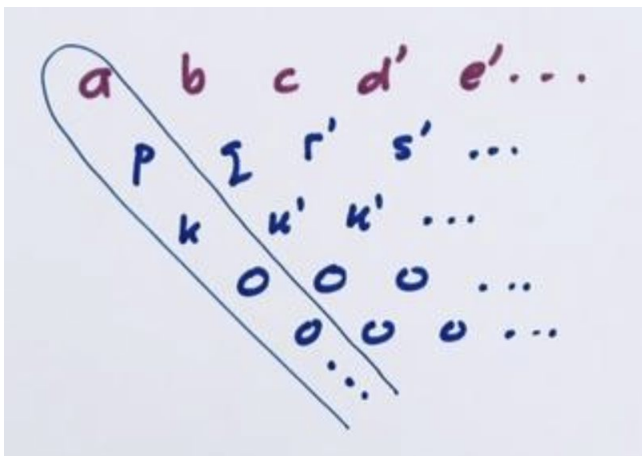


3. The difference tables of each of the sequences 1 and n and n^2 is zero from the fourth row onwards, and it follows that the difference table for $An^2 + Bn + C$ must be zero as well beyond the third row.



4. Thus our original sequence and the sequence from the quadratic expression $An^2 + Bn + C$ have matching leading diagonals: **a p k 0 0 0 …** . This means that all terms of the two difference tables—including the entries of the first row, the two sequences—match.

We conclude: *Our original sequence is generated by the formula* $An^2 + Bn + C$.



5. Could two different quadratic expressions $An^2 + Bn + C$ and $A'n^2 + B'n + C'$ generate the same sequence?

If this were possible, then their difference $\left(A - A'\right)n^2 + \left(B - B'\right)n + \left(C - C'\right)$ would be an expression that gives only zero values, that is, produces a sequence that is all zeros. But if any of the values $A - A'$, $B - B'$, or $C - C'$ is non-zero, this would not be the case. So it must be the case that $A = A'$, $B = B'$, and $C = C'$.

We conclude: *There is only one quadratic expression that generates a sequence with constant second differences.*

So once you have found one quadratic expression that works, you have found <u>the</u> quadratic expression that works!