WILD COOL MATH!

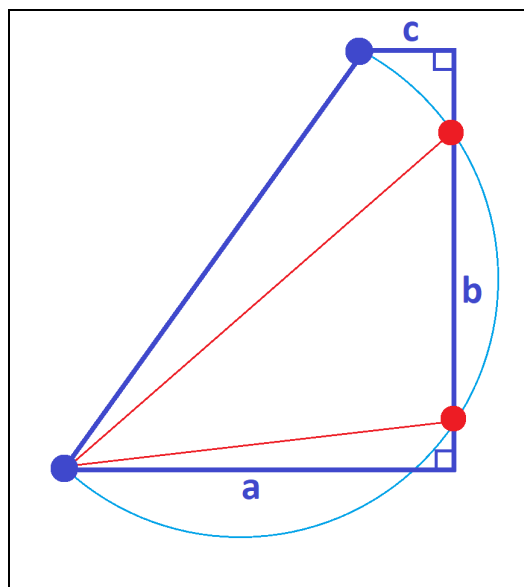## CURIOUS MATHEMATICS FOR FUN AND JOY

SEPTEMBER 2019

TODAY'S PUZZLER

On the diameter of a semi-circle a trapezoid with two horizontal sides and one vertical side of lengths a, b, and c is constructed as shown. The semicircle intersects the vertical side of length b twice.

Show that the slopes of the two red lines indicated are

$$\frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

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LILL'S METHOD

In 1867, Austrian engineer Eduard Lill presented stunning geometric representations of solutions to polynomial equations. In 2011, American mathematician Thomas Hull likened these pictures as hunting for solutions via hunting for turtles. And in 2019 math YouTuber *Mathologer* shared a lovely and accessible video describing Hull's approach to Lill's diagrams. (All references can be found on [Wikipedia](#) under "Lill's method.")

This month I'd like to add this essay as a supplement to the Mathologer's wonderful video, making clear some points presented in that video. This essay is fully self-contained, but do be sure to watch the video too!

Let's start by …

Solving the opening puzzler

Focus on just one of the red line segments.
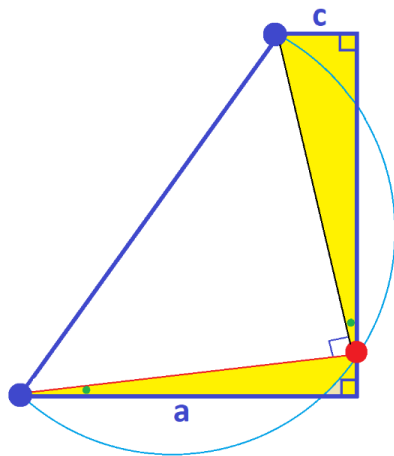
Draw a line segment from the opposite end of the semi-circle diameter to the red dot for that segment to make a triangle. (In my picture, this triangle is white). By Thales' theorem, any triangle constructed on the diameter of a circle is a right triangle (despite what my bad drawing seems to show!) Some angle chasing proves that the

two angles marked with a green dot have the same measure thus demonstrating that the two right triangles in yellow are similar.

We'll now use the following basic fact.

In a right triangle with identified base of length L and hypotenuse at slope m (assuming that the triangle is turned so that the base is horizontal), the length of the third side of the triangle—the "rise"—is mL.
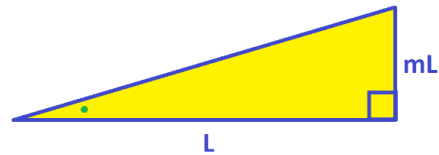
Suppose that the red line segment we are considering has slope m.
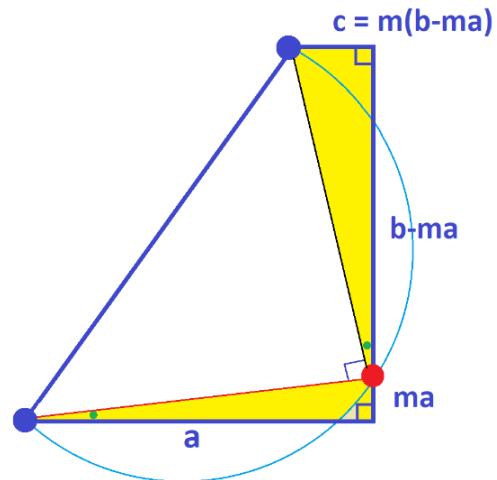
Then we see that the yellow triangle of base length a has rise ma, showing that the second yellow triangle has base $b - ma$ and consequently rise of $m(b - ma)$. (Similar right triangles have hypotenuses of the same slope.)

Consequently $c = m(b - ma)$.

This shows that m is a solution to the quadratic equation

$$am^2 - bm + c = 0$$

and so must be one of the two values
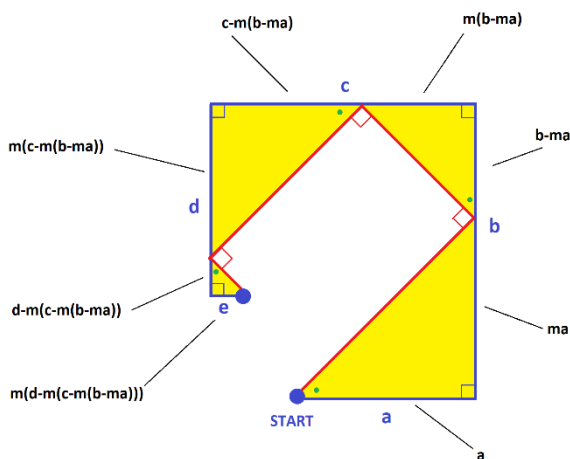
$$\frac{-(-b) \pm \sqrt{(-b)^2 - 4ac}}{2a}.$$

The second red line segment must have slope the second solution.

Pushing matters further

It's fun to chase through more complicated diagrams likes these.

In this picture of similar right triangles, if the first right triangle with a corner at start has hypotenuse of slope m, then we see that m is a solution to the equation

$$e - m\left(d - m\left(c - m\left(b - ma\right)\right)\right) = 0.$$

That is, m is a solution to

$$e - md + m^2c - m^3b + m^4a = 0.$$

The alternating plus/minus pattern to the coefficients of this polynomial can be eliminated by writing this as

$$a\left(-m\right)^4 + b\left(-m\right)^3 + c\left(-m\right)^2 + d\left(-m\right) + e = 0.$$

And so we have shown that $-m$ is a solution to the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$

This suggests a geometric method for finding solutions to polynomial equations!

To solve a quartic equation, say, $ax^4 + bx^3 + cx^2 + dx + e = 0$, *start somewhere in the plane and cycle moving east, then north, then west, then south, then east (that is, always turn $90°$ counterclockwise) the given amounts a, b, c, d, and e. Construct a right triangle nestled between the first two line segments, with one corner at start, and from it construct right triangles nestled between consecutive line segments with hypotenuses connected end-to-end also "turning" $90°$ from one to the next.*

If the first right triangle has just the right slope m so that the final right triangle in this construction has corner right at the endpoint of the final line segment, then $-m$ is a solution to the polynomial equation!

Polynomial equations of other degrees are solved this way too.

Negative coefficients?

Our discussion above seems to assume that the coefficients a, b, c, d, ... should be positive. (Move east a units, then move north b units, and so on.) What if some of these coefficients are negative?

Let's examine the geometric picture associated to the equation

$$x^3 + 2x^2 - x - 2 = 0.$$

This is the expanded version of the equation $(x - 1)(x + 1)(x + 2) = 0$ with solutions $x = 1$, $x = -1$, and $x = -2$.

Here $a = 1$, $b = 2$, $c = -1$, and $d = -2$.

So we start with a path of line segments constructed by moving 1 unit eastwards, 2 units northward, -1 units westward, and -2 units southward. Should negative distances correspond to motion in the reverse direction? Let's try it!

This diagram shows the four directions of linear motion in turn and the path in bold is the one that results in interpreting "negative distances forward" as motion backward.
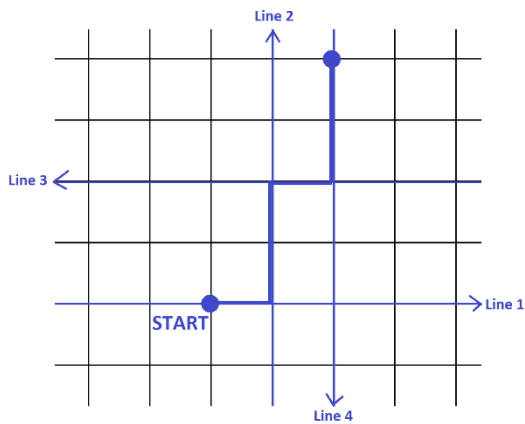
We know that $x = -1$ is a solution to the polynomial equation. What happens if we draw a right triangle between lines 1 and 2 with hypotenuse of slope 1?
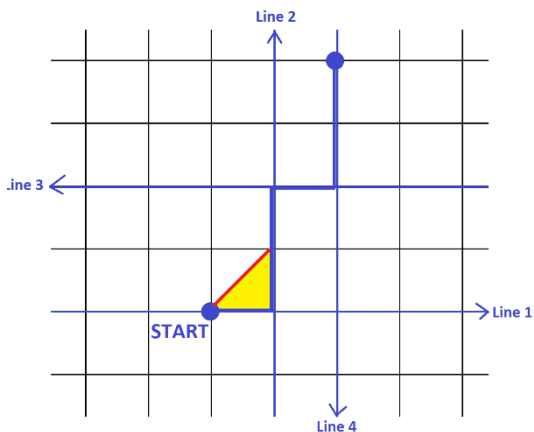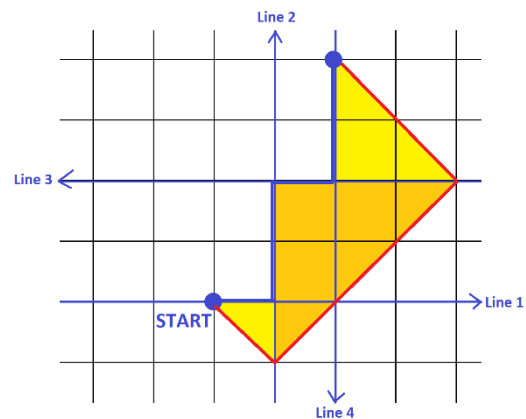
We now need to draw a right triangle between lines 2 and 3 with hypotenuse turned $90°$. There is only one such triangle.
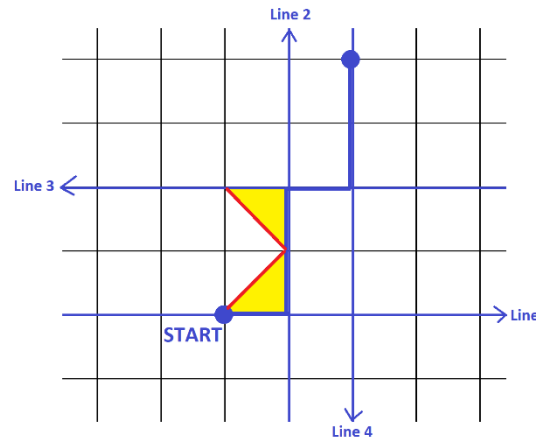
Now we need a right triangle between lines 3 and 4 with hypotenuse turned $90°$. There is only one such triangle. And, lo and behold, it "lands" right on the endpoint of the path.
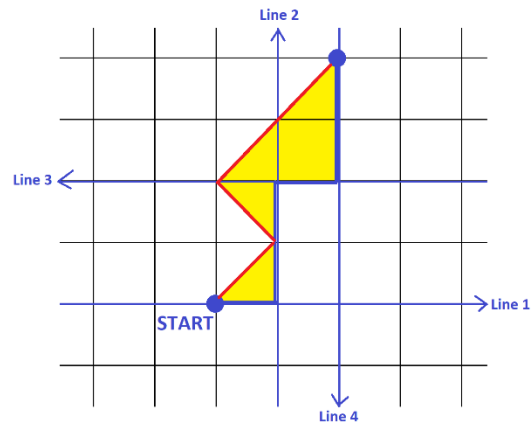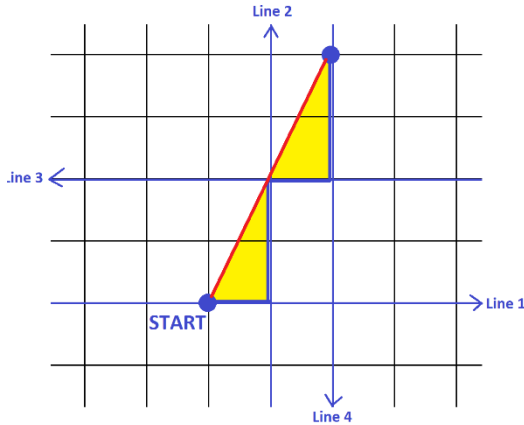
Here's the diagram for an initial right triangle with slope -1 for the solution $x = 1$.

And the diagram for an initial triangle of slope 2 (corresponding to the solution $x = -2$) contains a right triangle of zero

area. (Can you properly explain what we see here?)

So it seems that geometric approach works if we interpret negative coefficients as backward motion in this way.

Practice: Suppose a and b are positive numbers and c and d are negative numbers. In the diagram, suppose the initial right triangle has slope m.
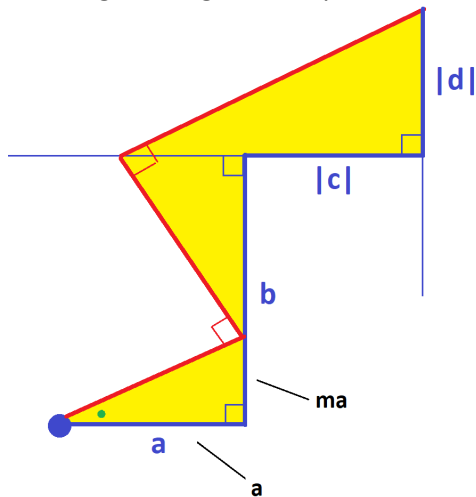
Is $-m$ indeed a solution to the equation $ax^3 + bx^2 + cx + d = 0$?

Challenge: Is it possible to see the solutions to $x^2 - 4 = 0$ via this geometric method? (How should one handle zero coefficients?) How about the solutions to $x^3 - 5x^2 + 6x = 0$? (Can one detect zero solutions?)
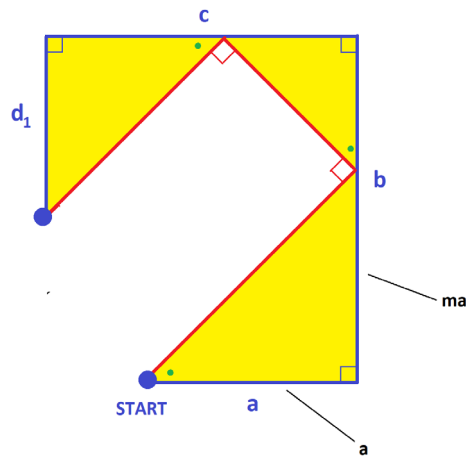
Challenge: Recall this diagram.
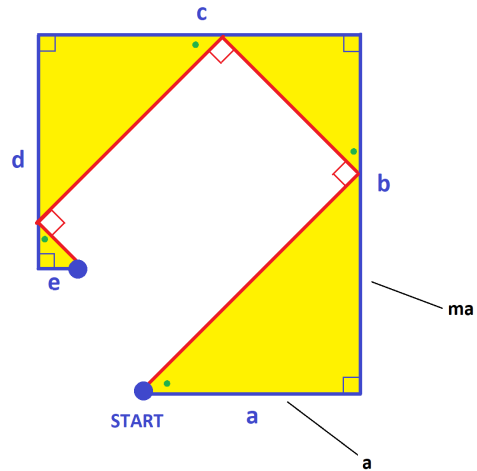
We deduced from it that $x = -m$ is a solution to

$$ax^4 + bx^3 + cx^2 + dx + e = 0 .$$

a) This next diagram shows that $x = -m$ is also a solution to which equation?

b) Develop a formal proof, by induction perhaps, that proves that a diagram of similar right triangles constructed along a set of line segments of lengths a, b, c, d, … (with negative and zero lengths interpreted appropriately) in the manner of this essay and having initial right triangle with hypotenuse of slope m gives $x = -m$ as a solution to the polynomial equation with coefficients a, b, c, d, … .

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POLYNOMIAL DIVISION

Is $ax^3 + bx^2 + cx + d$ a multiple of $x + m$?

Likely not. But let's see how close it is to being one.

Now the first term $ax^3 = ax^2 \cdot x$ of the polynomial is a multiple of x, which is close to being a multiple of $x + m$. Let's write

$$ax^3 + bx^2 + cx + d$$
$$= ax^2 \cdot (x + m) - max^2 + bx^2 + cx + d.$$
$$= ax^2 \cdot (x + m) + (b - ma)x^2 + cx + d.$$

The second term in this expression, $(b - ma)x \cdot x$ is a multiple of x, which is close to being a multiple of $x + m$. Let's write

$$ax^3 + bx^2 + cx + d$$
$$= ax^2 \cdot (x + m) + (b - ma)x \cdot (x + m)$$
$$+ (c - m(b - ma))x + d.$$

The third term is a multiple of x, which is close to being a multiple of $x + m$. Let's write

$$ax^3 + bx^2 + cx + d$$
$$= ax^2 \cdot (x + m) + (b - ma)x \cdot (x + m)$$
$$+ (c - m(b - ma)) \cdot (x + m)$$
$$+ (d - m(c - m(b - ma)))$$

This shows that $ax^3 + bx^2 + cx + d$ is very close to being a multiple of $x + m$: it is "off" only by the number
$$d - m(c - m(b - ma)),$$
that is, by the number
$$d + c(-m) + b(-m)^2 + a(-m)^3.$$

Now suppose this number is zero. That is, that $x = -m$ is a solution to
$$d + cx + bx^2 + ax^3 = 0.$$

Then we have just shown that in this case

$$ax^3 + bx^2 + cx + d$$
$$= (x + m)\left(ax^2 + (b - ma)x + (c - m(b - ma))\right)$$

Aside: This is the famous Factor Theorem in polynomial algebra. *If $x = -m$ is zero of a polynomial, then $x + m$ is a factor of the polynomial (and vice versa).*

But if $x = -m$ is indeed a solution to $d + cx + bx^2 + ax^3 = 0$, then we also have the following geometric diagram.
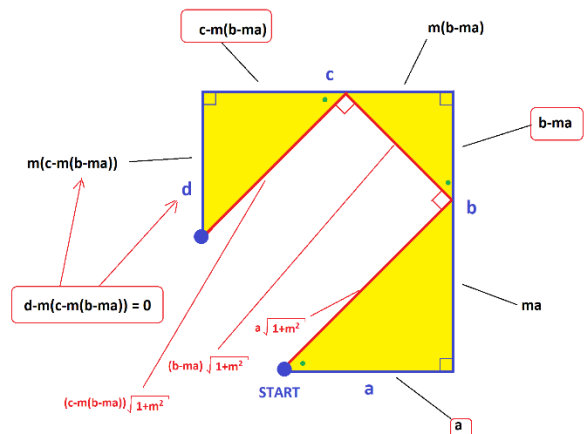
The red lines are segments of lengths a, $b - ma$, and $c - m(b - ma)$, but scaled up by a common factor of $\sqrt{1 + m^2}$. That is, the red segments represent the start of a diagram for finding zeros to
$$ax^2 + (b - ma)x + (c - m(b - ma)).$$
(That the diagram is scaled does not matter: the slope of the hypotenuse of a right triangle does not depend on the scale of the triangle.)

We have:

Given a polynomial p , if one set of right triangles shows that $x + m$ is a factor of p giving

$$p(x) = (x + m)q(x)$$

for some polynomial q , then the hypotenuses of those right triangles can be used to find a factor of q . And so on!

WHAT IF YOU MISS?

Lill's geometric construct is not, in practice, an easy way to find solutions to polynomial equations: How do guess an initial slope m for the hypotenuse to the initial right triangle that yields a sequence of triangles with final one landing with a corner right on the desired endpoint? You will most likely miss!

Challenge: Let p be the polynomial given by $p(x) = ax^3 + bx^2 + cx + d$.

a) In the diagram below, show that $r = p(-m)$.

b) In the diagram below show that $s = -p(-m)$.

RESEARCH CORNER

The quadratic equation $x^2 + 2x + 1 = 0$ has only one solution. Does Lill's method show this?

There are no real solutions to the quadratic equation $x^2 + x + 2 = 0$. Does Lill's method show this? Can a version of Lill's method be used to detect its complex number solutions?
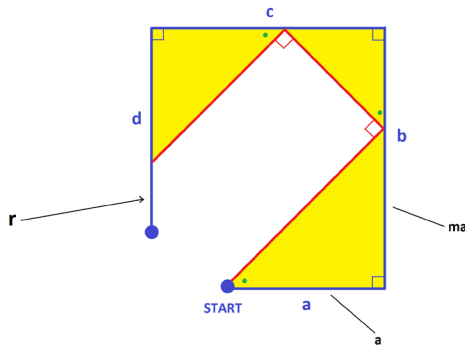
Can Lill's method be used to prove that if a cubic equation has two real solutions, then it must also have a third real solution? (Even if that third solution is a repeated root?)