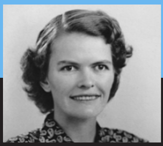
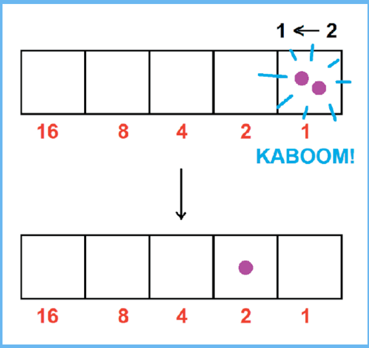
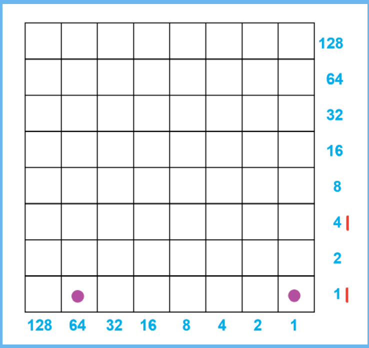
Julia Robinson Mathematics Festival®

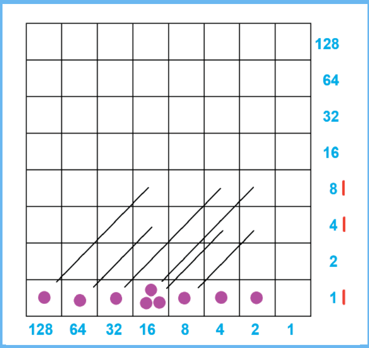
Exploding Dots

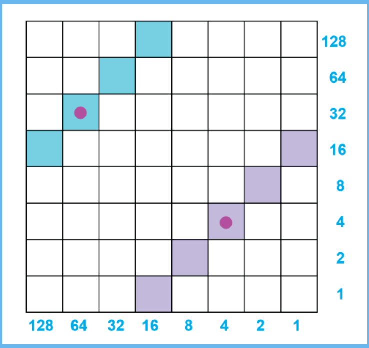


Julia Robinson
(1919 - 1985)









Exploding Dots is an astounding mathematical story that starts at the very beginning of mathematics – it assumes nothing – and swiftly takes you on a wondrous journey through grade school arithmetic, polynomial algebra, and infinite sums to unsolved problems baffling mathematicians to this day. Visit **globalmathproject.org** to join an exploding community and to learn more about the Global Math Project.

Visit **www.JRMF.org** for more information about Julia Robinson Mathematics Festivals.

Activities by James Tanton, Founder, Global Math Project. Edited by Mark Saul, Executive Director, Julia Robinson Mathematics Festival, and Alice Peters.

SETTING THE SCENE: GRAPE CODES

Consider a row of dishes extending from right to left as far as we want, each labeled with a consecutive power of two, in order, starting with 2^0. In the picture below there are six dishes.

Question 1: ***If I have ten dishes, what would be the label of the leftmost dish?***

We drop any number of grapes into any of the dishes. Each grape has the value given by the label of the dish in which it sits. Then we add the values of the grapes. For example, three grapes in the dish labeled 8 and two in the dish labeled 1 together have a total value of 8+8+8+1+1 = 26. We will write 3|0|0|2 as a code for this arrangement of grapes, whose value is twenty-six. (We ignore all leading zeros; that is, we won't record the empty dishes to the left of the leftmost non-empty dish.)

3|0|1|0

6|0|2

1|1|0|1|0

26

Question 2: ***Other "grape codes" for twenty-six are possible. Four more grape codes for the number twenty-six are shown in the illustration above.***

In fact, if we look hard, we can find a total of 114 different grape codes for the number twenty-six. That is, there are 114 different ways to represent the number twenty-six with grapes in dishes (but we need not actually list them all here).

(a) Of these codes shown above, is the last one ("26") the code that uses the most grapes? Is 1|1|0|1|0 the code that uses the fewest grapes? How would you know?

(b) (Difficult) Are there two different codes for twenty-six that use the same number of grapes? Can you find five different codes that use the same number of grapes?

Question 3: ***What are the values of the following 'grape codes'?***

(a) 2|1|1|0 **(b)** 7|0|0 **(c)** 1|1|1|1 **(d)** 2|2|2|2|2 **(e)** 1|0 **(f)** 1|0|0|1

Question 4: ***Find at least two grape codes for each of the following numbers. Are there any numbers for which only one grape code is possible?***

(a) 12 **(b)** 6 **(c)** 3 **(d)** 1 **(e)** 24 **(f)** 25 **(g)** 29

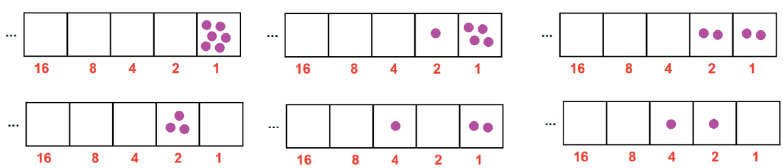
Question 5: ***Using only two grapes, code at least 6 different numbers, all of which are greater than 10.***

Question 6: ***Suppose we list all the numbers that can be coded using two grapes. The list begins with:***

2, 3, 4, 5, 6, 8, 9, 10 . . .

What is the 50th number on this list?

Question 7: ***There are 6 different grape codes for the number six:***



(a) Show that there are also 6 grape codes for the number seven. (Hint: can you use the six codes above to do this?) Draw diagrams for each of the codes. Find some ways in which the set of diagrams for seven differs from the set of diagrams for six.

(b) Is it true in general that the count of grape codes for an odd number is equal to the count of grape codes for the even number just before it?

(c) Is it true in general that the count of grape codes for an even number is equal to the count of grape codes for the odd number just before it?

The table shows the number of different grape codes for the first few even numbers.

Number	2	4	6	8	10	12	14	16	26
# of grape codes	2	4	6	10	14	?	?	?	114

(d) Fill in the three missing entries. Care to find a few more entries?

(e) Find some patterns in the sequence of numbers you are generating:

2, 4, 6, 10, 14 . . .

Can you be sure any patterns you see will continue?

Now suppose we have six dishes, labeled as before. A code for a number with at most one grape in each dish is called a binary code for that number. For instance, 1|1|0|1|0 is a binary code for the number twenty-six and 1|1|0 is a binary code for the number six. On the other hand, 2|1|0 is not a binary code for any number, because in a binary code, no dish can contain more than one grape.

Question 8: *Find a binary code for the number one hundred.*

(a) Can you be sure that every positive integer has a binary code?

(b) Could a positive integer have two different binary codes?

(Questions (b) and (c) above are not so simple. We will address problem (b) in the next session. But what are your thoughts right now?)

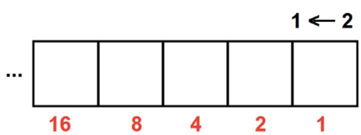
THE 1 ← 2 MACHINE

Question 9: ***The binary code for the number five is 1|0|1. If we put one more grape into the ones dish, the overall grape value is six and the grape code is 1|0|2. How can you rearrange the grapes so that the value is still six, but you now have a binary code? That is, how can you adjust the representation of the number six so that there is no more than one grape in each dish?***

Question 10: ***Find the binary code for the number seven. Add one grape to the ones dish to make the new value equal to eight. How can you rearrange the grapes to represent the same value as a binary code***?

Question 11: ***More generally, suppose you have the binary code for the number N. How could you construct the binary code for the number N+1?***

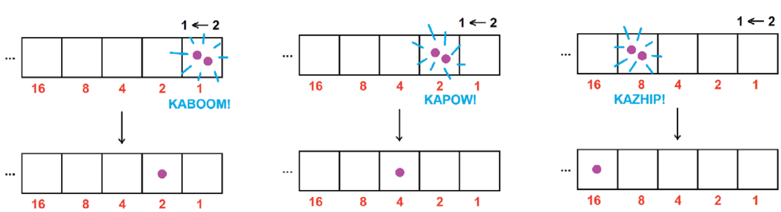
In the story *Exploding Dots* from the Global Math Project (see **www.gdaymath.com/courses/exploding-dots/**), our row of dishes becomes a "two-one machine," written "1 ← 2."



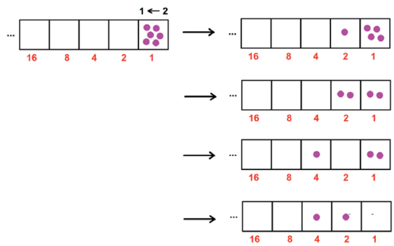
One puts dots (or grapes) into the rightmost box and lets them "explode" in the following way:

> *Whenever there are two dots in a box, any box, they explode and disappear–KAPOW!–to be replaced by one dot located in the box to the left.*

And indeed, two dots in any one box have the same combined value as one dot just to their left.



In this way, placing a number of dots in the rightmost box eventually generates a representation of that number with at most one dot in each box. This shows that every positive integer has at least one binary code. For example, placing 6 dots into the machine eventually gives the binary code 1|1|0 for the number 6.



Question 12: **(a)** ***What numbers have the following binary codes?***

(i) 1|0|1|1|0 **(ii)** 1|0|0|1|0|0|1|0|0 **(iii)** 1|1|1|1|1 **(iv)** 1|1|1|1|1|0

(b) Find binary codes for the first 20 positive integers. What do you notice about the codes for the even numbers? The codes for the odd numbers?

b) Anouk says she invented a divisibility rule for the number 4:

A number is divisible by 4 precisely when its binary code ends with two zeros.

Do you agree with her rule?

(c) (Difficult!) Can you devise a divisibility rule for the number 3 based on the binary codes of numbers?

Question 13: ***Aba has a curious technique for finding the binary code of a number.*** She writes the number at the right of a page and halves it, writing the answer one place to its left, ignoring any fractions if the number was odd. She then repeats this process until she gets the number 1 . Then she writes 1 under each odd number she sees and 0 under each even number. The result is the binary code of the original number.

Here's her work for computing the binary code for 22:

Why does her technique work?

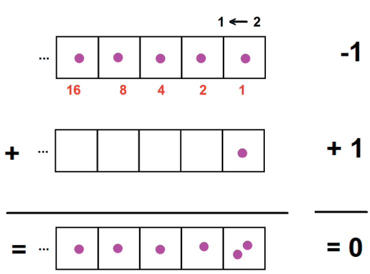
1	2	5	11	22
1	0	1	1	0

Question 14: ***Here's a fun way to compute the product of two numbers, say, 22 × 13:*** Write the two numbers at the head of two columns, halve the left number (ignoring fractions) and double the right number, and repeat until the number 1 appears in the left-hand column. Then cross out all the rows that have an even number on the left, and add all the numbers on the right that survive. That sum is the answer to the original product!

$$\begin{array}{rcr} \cancel{22} & \cancel{\mathbf{x}} & \cancel{13} \\ 11 & \mathbf{x} & 26 \\ 5 & \mathbf{x} & 52 \\ \cancel{2} & \cancel{\mathbf{x}} & \cancel{104} \\ 1 & \mathbf{x} & \underline{208} \\ & & 286 \end{array}$$

Why does this method work?

Question 15: ***Allistaire suggested that the binary code for -1 should be . . . 1|1|1|1|1|1|1; that is, an infinitely long string of ones going infinitely far to the left. He argued that you can check this by placing this infinite string in a 1 ← 2 machine, then adding a single dot in the rightmost box. This produces, after explosions, an empty diagram: zero.***

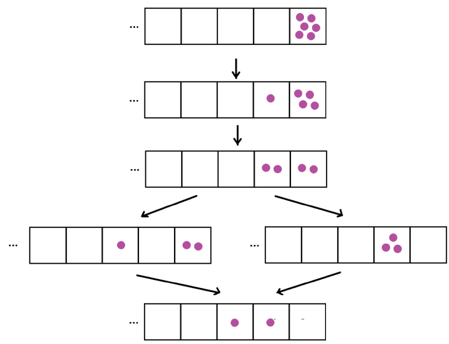


This showed, Allistaire argued, that (-1) + (+1) = 0 if we use this way of coding the number -1.

Do you agree?

GRAPE CODES AND BINARY CODES

The following diagram shows all the choices one can make when performing explosions on 6 dots to lead to the binary code 1|1|0 for the number 6. The diagram also shows all 6 ways we can represent 6 using grapes!

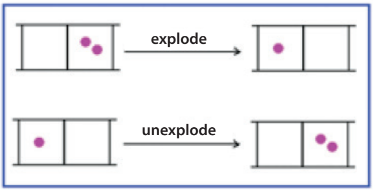


Question 16: **(a)** ***Draw an analogous diagram for 12 dots placed in a 1 ← 2 machine. Show all the choices one can make for explosions and show that all paths lead to the same final binary code 1|1|0|0.***

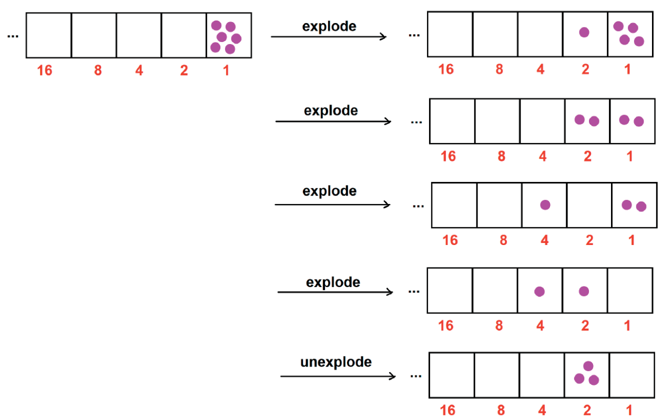
(b) There are 20 ways to represent the number 12 with grapes in dishes. Do all 20 grape codes appear in your diagram? Do all paths lead to the same binary code for 12?

(c) In general, when one draws a diagram of all possible explosions for n dots placed in a 1 ← 2 machine, is the diagram sure to contain all the possible grape codes for N? Do all paths lead to the same final binary code for N?

Just as we can 'explode' dots, we can 'unexplode' them as well: one dot in a particular box unexplodes to create two dots in the box just to the right:



Question 17: ***Starting with six dots in a 1 ← 2 machine, one can perform a sequence of five explosions and "unexplosions" that produces all 6 codes for the number six in terms of grapes.***



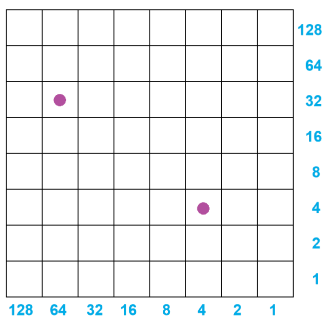
(a) Starting with 1 dot in the 1 ← 2 machine, is there a sequence of 19 explosions and unexplosions that takes one through all 20 possible codes for 12 in terms of grapes?

(b) Actually, prove that for each positive integer N, there is a sequence of explosions and unexplosions one can perform—starting with N dots in the rightmost box of a 1 ← 2 machine—to pass through all the possible grape codes of N without repeating a code.

Comment: The 2018 ARML power question at www.arml.com also explores these questions about codes for numbers, but not in the language of grapes or of Exploding Dots.

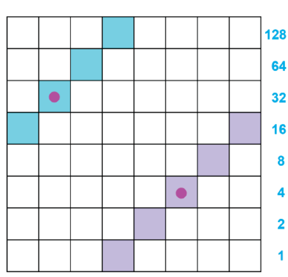
NAPIER'S MULTIPLICATION CHECKERBOARD

Five centuries ago, Scottish mathematician John Napier (1550 – 1617), best known for his invention of logarithms, suggested working with a two-dimensional array of boxes, with each row and each column labeled with a power of two.
A dot (or a grape or a pebble) in any box is given the value of the product of its column and row numbers. For example, one dot in this picture has the value $64 \times 32 = 2048$ and the other has the value $4 \times 4 = 16$. Together they represent the number $2048 + 16 = 2064$. One can thus represent very big numbers on this two-dimensional array.



Napier noted that you can slide a dot anywhere on the southwest diagonal on which it sits and not change its value and hence not change the total value of several dots on in the grid.

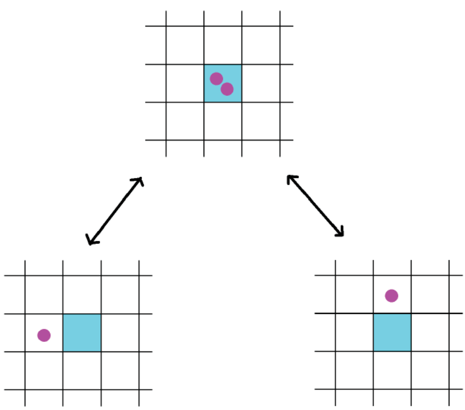
Question 18: ***Explain why a single dot in any of the light blue cells will have the same value. Explain why a single dot in any of the light purple cells will have the same value.***



Napier also noted that each row of the table is operating as its own machine, as is each column!

Any two dots in the same cell can be erased—they explode, "kaboom"—and can be replaced either by one dot placed one cell to their left, or by one dot placed one cell above them, your choice!

One can also unexplode dots:

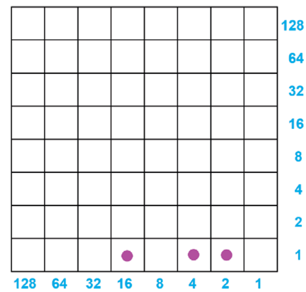


By placing dots in the grid we can represent large numbers, and by performing slides, explosions, and unexplosions, we can change the representations of those numbers in lots of different—but always equivalent—ways. And with this power, Napier realized we can perform some sophisticated arithmetic!

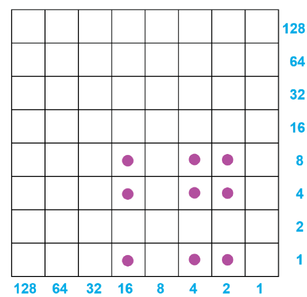
Make sure you actually try what follows with a checkerboard or hand-drawn grid of squares using pennies or counters, or grapes! The grid does not have to be 8 by 8: any size will do. People in the 1600s used a square sheet of cloth marked into squares and beads for counters.

MULTIPLICATION

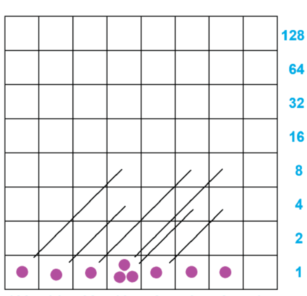
As we saw earlier, the binary code for twenty-two is 1|0|1|1|0. Here is the number twenty-two represented in Napier's checkerboard as $16 \times 1 + 4 \times 1 + 2 \times 1$. (Since the bottom row of the checkerboard is its own $1 \leftarrow 2$ machine, one could place 22 dots into the right corner box and perform explosions in just the bottom row to get this binary code.)

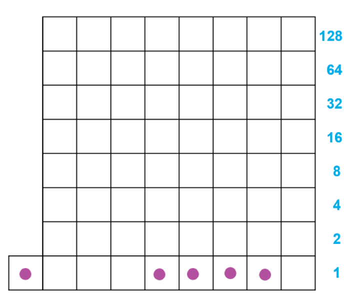


Now here now is 22×8 plus 22×4 plus 22×1 in Napier's checkerboard, that is, here is 22×13.



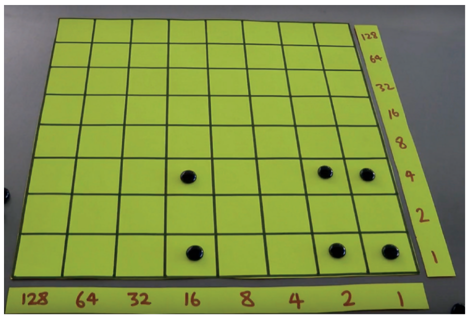
To see what the value of this product is, slide all the dots diagonally down to the bottom row, perform explosions along the bottom row, and read off the answer! We see, after adding an extra box, that 22×13 is $256 + 16 + 8 + 4 + 2 = 256 + (16 + 4) + (8 + 2) = 286$.





Question 19:

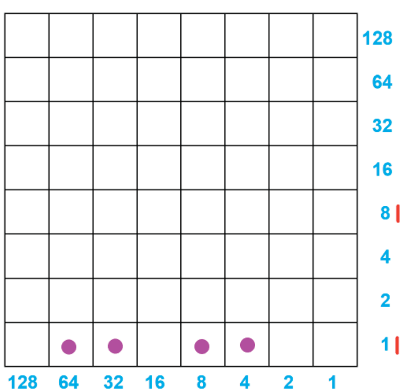
(a) ***What product is represented in this homemade checkerboard? What is the usual representation of this number?***



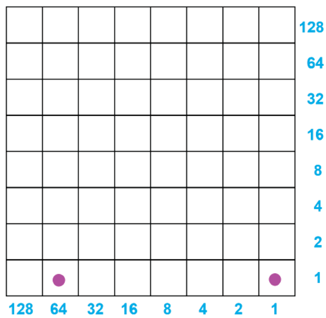
(b) Compute 51×42 with a checkerboard.

(c) Keep computing different products via Napier's method. Have some fun!

Question 20: **(a)** ***Here is the result of multiplying some number by 9 with Napier's checkerboard. What number was multiplied by 9? Can you see the answer by sliding dots to recreate the original multiplication problem?***



(b) The result of multiplying some number by 5 is 65, as shown. Can you recreate the original multiplication problem by unexploding and sliding dots?



(c) The previous two problems ask us to reverse the multiplication process. That is, they are each asking us to answer a division problem. (They first had us compute 108 ÷ 9 and the second 65 ÷ 5.) Can you compute 247 ÷ 13 on Napier's checkerboard? Can you compute 250 ÷ 13 and find the remainder of 3?

Develop a general technique for performing long division on Napier's checkerboard.

(d) Can you use Napier's checkerboard to find a number which, when multiplied by itself, gives the answer 196? What do you see if you follow the same checkerboard technique to try to find the square root of 200?

Comment: See Experience 11 of **www.gdaymath.com/courses/exploding-dots** for more discussion of Napier's checkerboard. (There we also discuss how Napier suggested performing addition and subtraction on his checkerboard too, essentially using only the bottom row of the checkerboard; that is, using only a single 1 ← 2 machine.)

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