Uplifting Mathematics for All

Exploding Dots™

HANDOUTS

Experience 6:

All Bases, All at once - Polynomials

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Handout A: *Division in Any Base*

The computations $276 \div 12$ and $(2x^2 + 7x + 6) \div (x + 2)$ are identical!

In a 1←10 machine.

$$276 \div 12$$
$$= 23$$

SAME PICTURE!

In a 1←x machine.

$$(2x^2 + 7x + 6) \div (x + 2)$$
$$= 2x + 3$$

Here are some practice problems for you to try, if you like:

1. a) Compute $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1)$.

   b) Compute $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1)$.

   If I tell you that x is actually 10 in both these problems what two division problems in ordinary arithmetic have you just computed?

2. Here is a polynomial division problem written in fraction notation. Can you do it? (Is there something tricky to watch out for?)

$$\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$$

3. Show that $(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1)$ equals $x^3 + 3x^2 + 3x + 1$.

   a) What is this saying for $x = 10$?
   b) What is this saying for $x = 2$?
   c) What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11?
   d) What is this saying for $x = 0$?
   e) What is this saying for $x = -1$?

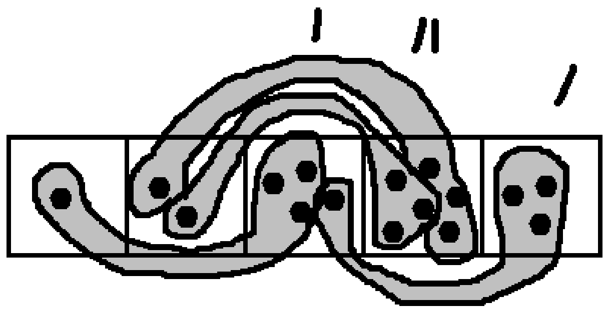
Solutions to Handout A

1.
   a) $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1) = \; x^3 + x^2 + 2x + 1$

   b) $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1) = x^2 + 2x + 3$

And if x happens to be 10, we've just computed 23541 ÷ 21 = 1121 and 13653 ÷ 111 = 123.

2. We can do it. The answer is $x^2 + 2x + 1$.



3.
   a) For $x = 10$ it says 14641 ÷ 11 = 1331

   b) For $x = 2$ it says 81 ÷ 3 = 27

   c) For $x = 3$ it says 256 ÷ 4 = 64
   For $x = 4$ it says 625 ÷ 5 = 125
   For $x = 5$ it says 1296 ÷ 6 = 216
   For $x = 6$ it says 2401 ÷ 7 = 343
   For $x = 7$ it says 4096 ÷ 8 = 512
   For $x = 8$ it says 6561 ÷ 9 = 729
   For $x = 9$ it says 10000 ÷ 10 = 1000
   For $x = 11$ it says 20736 ÷ 12 =1728

   d) For $x = 0$ it says 1 ÷ 1 = 1.

   e) For $x = -1$ it says 0 ÷ 0 = 0. Hmm! That's fishy! (Can you have a 1 ← 0 machine?)

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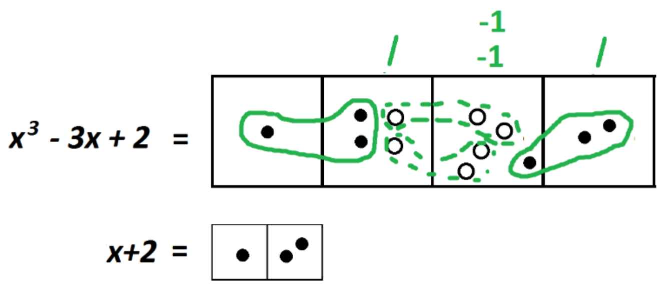
Handout B: *A Problem and Resolution*

We can even work with antidots in polynomial division.



Here are some practice problems for you to try, if you like.

1. Compute $\frac{x^3-3x^2+3x-1}{x-1}$.

2. Try $\frac{4x^3-14x^2+14x-3}{2x-3}$.

3. If you can do this problem, $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1}$, you can probably do any problem!

4. This one is crazy fun: $\frac{x^{10}-1}{x^2-1}$.

   Aside: Is there a way to conduct the dots and boxes approach with ease on paper? Rather than draw boxes and dots, can one work with tables of numbers that keep track of coefficients? (The word *synthetic* is often used for algorithms one creates that are a step or two removed from that actual process at hand.)

5. Can you deduce what the answer to $(2x^2+7x+7) \div (x+2)$ is going to be before doing it?

6. Compute $\frac{x^4}{x^2-3}$.

7. Try this crazy one: $\frac{5x^5-2x^4+x^3-x^2+7}{x^3-4x+1}$.

   If you do it with paper and pencil, you will find yourself trying to draw 84 dots at some point. Is it swift and easy just to write the number "84"? In fact, how about just writing numbers and not bother drawing any dots at all?

Solutions to Handout B

1. $\frac{x^3-3x^2+3x-1}{x-1} = x^2 - 2x + 1$.

2. $\frac{4x^3-14x^2+14x-3}{2x-3} = 2x^2 - 4x + 1$.

3. $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1} = 4x^3 + 2x^2 + 5x - 1$.

4. $\frac{x^{10}-1}{x^2-1} = x^8 + x^6 + x^4 + x^2 + 1$.

5. We know that $(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$, so I bet $(2x^2 + 7x + 7) \div (x + 2)$ turns out to be $2x + 3 + \frac{1}{x+2}$. Does it? Can you make sense of remainders?

6. $\frac{x^4}{x^2-3} = x^2 + 3 + \frac{9}{x^2-3}$.

7. $5x^2 - 2x + 21 + \frac{-14x^2+86x-14}{x^3-4x+1}$.

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Handout C: *WILD EXPLORATIONS*

Here are some "big question" investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: CAN WE EXPLAIN AN ARITHMETIC TRICK?

Here's an unusual way to divide by nine.

To compute 21203 ÷ 9, take the digits in "21203" from left to right computing the partial sums along the way as follows

$$\begin{array}{ll} \mathbf{2} & \mathbf{= 2} \\ \mathbf{2+1} & \mathbf{= 3} \\ \mathbf{2+1+2} & \mathbf{= 5} \\ \mathbf{2+1+2+0} & \mathbf{= 5} \\ \mathbf{2+1+2+0+3} & \mathbf{= 8} \end{array}$$

and then read off the answer

$$21203 \div 9 = 2355 \ R \ 8 .$$

In the same way,

$$1033 \div 9 = 1 \mid 1+0 \ \mid 1+0+3 \mid \quad R \quad 1+0+3+3 \quad = 114 \ R \ 7$$

and

$$2222 \div 9 = 246 \ R \ 8$$

Can you explain why this trick works?

Here's the approach I might take: For the first example, draw a picture of 21203 in a $1 \leftarrow 10$ machine, but think of nine as $10 - 1$. That is, look for copies of $\boxed{\bullet \mid \circ}$ in the picture.

EXPLORATION 2: CAN WE EXPLORE NUMBER THEORY?

Use an $1 \leftarrow x$ machine to compute each of the following

a) $\frac{x^2-1}{x-1}$ b) $\frac{x^3-1}{x-1}$ c) $\frac{x^6-1}{x-1}$ d) $\frac{x^{10}-1}{x-1}$

Can you now see that $\frac{x^{number}-1}{x-1}$ will always have a nice answer without a remainder?

Another way of saying this is that

$$x^{number} - 1 = (x-1) \times (\text{something}).$$

For example, you might have seen from part c) that $x^6 - 1 = (x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$. This means we can say, for example, that $17^6 - 1$is sure to be a multiple of 16! How? Just choose $x = 17$ in this formula to get

$$17^6 - 1 = (17-1) \times (\text{something}) = 16 \times (\text{something}).$$

a) Explain why $999^{100} - 1$ must be a multiple of 998.

b) Can you explain why $2^{100} - 1$ must be a multiple of 3, and a multiple of 15, and a multiple of 31 and a multiple of 1023? (Hint: $2^{100} = (2^2)^{50} = 4^{50}$, and so on.)

c) Is $x^{number} - 1$ always a multiple of $x + 1$? Sometimes, at least?

d) The number $2^{100} + 1$ is not prime. It is a multiple of 17. Can you see how to prove this?