MATHEMATICAL ASSOCIATION OF AMERICA

CURRICULUM INSPIRATIONS

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GLOBAL MATH PROJECT

Uplifting Mathematics for All

WILDLY COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

OCTOBER 2019

IT'S TIME TO CELEBRATE GLOBAL MATH!
October 10 – 17 is Global Math Week from the Global Math Project, among *many* other global initiatives!

* Maths Week Scotland: Sept 30 - Oct 6
https://www.mathsweek.scot/
World Maths Day Challenge: Oct 10 - 15: http://www.primegrid.com/forum_thread.php?id=8793#132794
* Irish Maths Week: Oct 12 - 20
https://www.mathsweek.ie/2019/
* World Maths Day: Oct 15 (UNICEF)
https://en.wikipedia.org/wiki/World_Maths_Day

And our GMP friends at BUBBY MATHS and AIMSSEC are also hosting a 30-minute Global Maths Lesson on October 10! The topic is **FANTASTIC FRACTALS**.

Check it out and join the fun!
https://www.bubblymaths.co.uk/gml/

This special essay offers a nod to self-similarity: the full-on self-similarity of fractals and the self-similarly that can appear in number sequences from *EXPLODING DOTS* (or its grape version!).

It is written as its own self-contained experience.

So try the Fantastic Fractals Global Maths Lesson as a start to Global Math Week – October 10th—and perhaps also try this lesson as another start to Global Math Week, and also go to

www.globalmathproject.org

for the full EXPLODING DOTS experience to try that week too–and all year long!

So much mighty fine math to explore!

Enjoy!

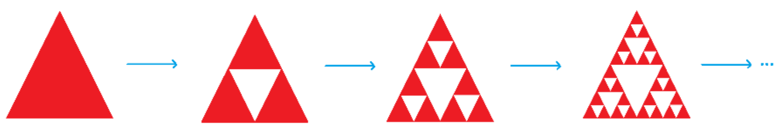
Flyer by Sophia Gerrik-Stier

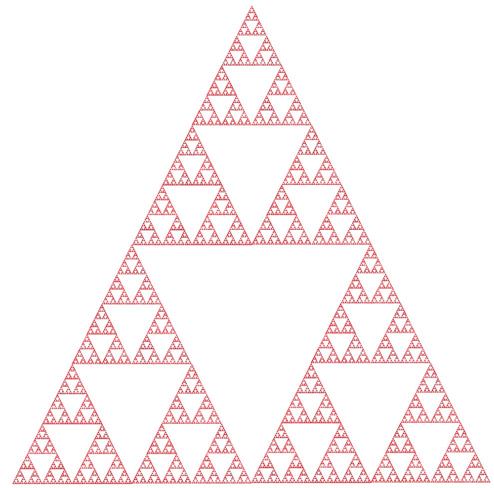
1. FRACTALS

A **fractal**, loosely speaking, is a structure composed entirely of smaller copies of itself!

Perhaps the most famous example of a fractal is the Sierpinski triangle first described in 1915 by Polish mathematician Waclaw Sierpinski.

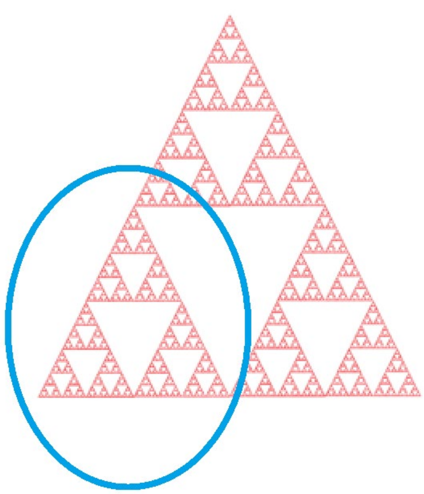
One starts with a triangle and removes its middle. Then one removes the middles of the three smaller triangles that remain. And then middles of the nine even smaller triangles that remain. And then middles of the twenty-seven even smaller triangles that remain. And so on. FOREVER!





Of course, we humans can never accomplish a task that goes on for all time, and so we will never see a real fractal in our human experience. But we certainly feel we can imagine one.

And when we do have an ideal Sierpinski triangle in our minds, we realise that it is a fractal: the entire object is composed of three smaller copies of its whole self.



The start to a giant Sierpinski tetrahedron.
www.bubblymaths.co.uk

2. NUMBER SEQUENCES WITH FRACTAL-LIKE FEATURES

Other "structures" can have fractal-like properties too: at least contain copies of themselves within themselves. Consider, for instance, this sequence of numbers.

1 1 2 1 2 3 1 2 3 4 1 2 3 4 5 1 2 3 4 5 6 …

Do you see the pattern to it? Can you say what the next seven terms shall be? And then the next eight?

This sequence has two surprising fractal properties.

Property 1: Cross out the first 1 in the sequence. Cross out the first 2 in the sequence. Cross out the first 3 in the sequence. And so on. What is left behind is the original sequence!

~~1~~ 1 ~~2~~ 1 2 ~~3~~ 1 2 3 ~~4~~ 1 2 3 4 ~~5~~ 1 2 3 4 5 ~~6~~ …

Property 2: Cross out all the 1s. What is left behind is a copy of the original sequence, but with each entry increased by one.

~~1~~ ~~1~~ 2 ~~1~~ 2 3 ~~1~~ 2 3 4 ~~1~~ 2 3 4 5 ~~1~~ 2 3 4 5 6 …

Question 1: *Describe what is left if you cross out all the 1s and 2s from the original sequence.*

Question 2: *Consider this sequence*

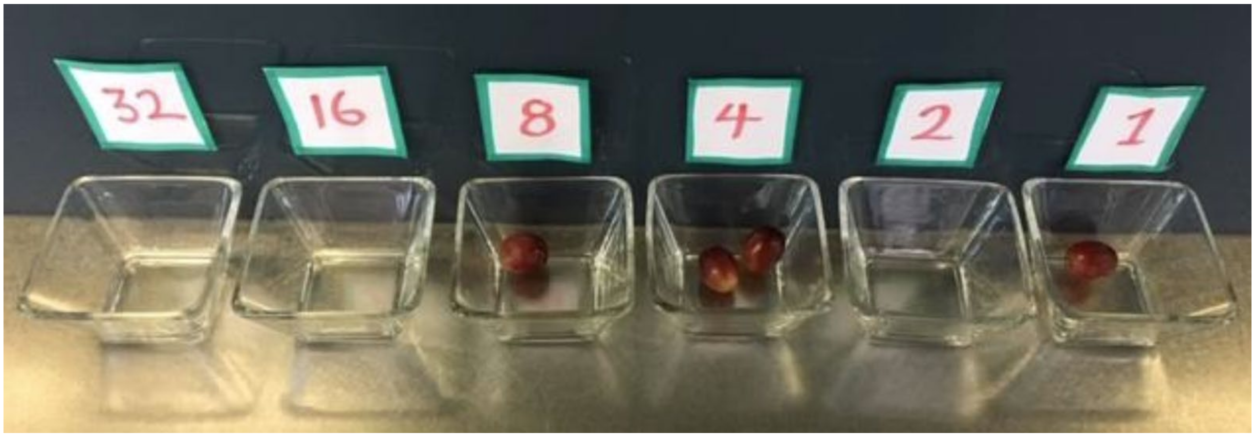
1 1 1 2 1 2 1 2 3 1 2 3 1 2 3 4 1 2 3 4 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 6 …

a) *Do you see a pattern to it? If so, what are the next seven terms of the sequence?*
b) *Cross out the first 1 and the first 2 and the first 3 and the first 4 and so on from the sequence. What is left behind?*
c) *Instead, cross out all the 1s. What is left behind?*

Question 3: *Can you construct a third example of a sequence of numbers possessing properties 1 and 2?*

3. A GRAPE-CODE NUMBER SEQUENCE

Consider a row of dishes extending as far to the left as ever we please. (I have just six dishes in the photo, but imagine more!) Label the dishes with the doubling numbers 1, 2, 4, 8, 16, 32, 64, 128, … starting at the right and heading to the left.



Question 4: *If I had ten dishes what would be the label of the leftmost dish?*

Each dish can hold grapes. And let's say that a grape in one dish has value the label of that dish. For example, in the picture above we have one grape of value 8, two each of value 4, none of value 2, and one of value 1. The total value of these grapes is $8+4+4+1=17$. We'll also denote this particular distribution of grapes as $1|2|0|1$ and call this a *grape-code* for the number seventeen. Other grape codes for seventeen are also possible: $3|2|1$ and $2|0|0|0|1$ and $8|1$, for instance.

Note: We are following a convention of not recording 0s for the empty dishes to the left of the leftmost non-empty dish.

Question 5: *Which number has grape code* $1|0|1|2|13|0$*? Which number has grape code* $1|0|0|0|0|0|0|0|0|0|0|0|0$*?*

Question 6: *Find three more grape codes for the number seventeen.*
(**Optional Challenge:** *List* all *the possible grape-codes for this number*!)

Question 7: *The code* $1|2|0|1$ *for seventeen uses four grapes. The code* $8|1$ *for seventeen uses nine grapes. Which code for seventeen uses the largest count of grapes? Which code for seventeen uses the least count of grapes? How do you know?*

Question 8: *Toni was playing with grape codes but was using very small dishes: each of her dishes could only hold 0 or 1 grapes. She found the following code for the seventeen*

$$1 \mid 0 \mid 0 \mid 0 \mid 1.$$

But she is wondering if there is another code she missed?

a) *Is this the only code for the number seventeen using at most one grape per dish? If so, how do you know? If not, what additional codes for seventeen is she missing?*

b) *Toni worked out codes for other numbers too, with at most one grape per dish. She noticed something about the codes she tried:*

All the codes for an odd number end with a 1.
All the codes for an even number end with a 0.

Is this observation sure to be true? How might you convince Toni that the code of an odd number, with at most one grape per dish, is sure to have 1 grape in the rightmost dish? Why can't an even number have a code with 1 grape in the rightmost dish?

BINARY CODES

Toni has accidentally stumbled upon a concept the world knows as *binary numbers:* each number can be represented as a code of 0s and 1s. Computers, which are based on electrical switches that are either "on" or "off," uses binary codes to represent numbers. To learn all about the power and fun of binary numbers, and other base systems, check out the astounding story of **EXPLODING DOTS** at www.gdaymath.com/courses.

Question 9: *Caroline was playing with grape codes too, using dishes that were also a little small, but this time each of her dishes could hold 0, 1, or 2 grapes.*

Caroline found the following codes for the number seventeen using at most two grapes per dish.

$$1 \mid 0 \mid 0 \mid 0 \mid 1 \qquad 1 \mid 2 \mid 0 \mid 1 \qquad 1 \mid 1 \mid 2 \mid 1$$

But she is not sure if she is missing some codes.

a) *Is her list complete? If not, which additional codes is she missing?*

b) *Each code for seventeen ends with a 1. Must each code for an odd number end with a 1?*

Caroline started making a table for the number of difference grape codes each number has using only 0, 1, or 2 grapes per dish.

Number	0	1	2	3	4	5	6	7	8	9	10	11	12
Count of codes	1	1	2	1	3	2	3	1	4	3	5	2	5

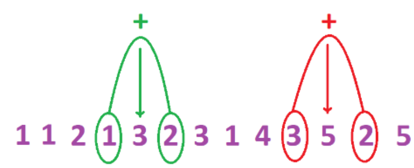
c) *What do you think of her starting the table with the number zero? How many grape codes are there for zero?*

d) *Do you care to add a few more entries to the table? Care to go all the way up to seventeen at least?*

e) *Do you see that* **1 1 2 1 3 2 3 1 4 3 5 2 5 …** *contains a copy of itself hidden within itself? (***Challenge:** *Can you explain why it satisfies what you observe?)*

THE STERN-BROCOT SEQUENCE

Caroline has accidentally stumbled upon here a sequence famous among mathematicians: the Stern-Brocot sequence. It has an astounding number of properties. For example, every second number in the sequence is the sum of its two neighbours. Also, if



we make all the fractions possible with numerator one term in the sequence and denominator the next in the sequence we get the following list of fractions: $\frac{1}{1},\frac{1}{2},\frac{2}{1},\frac{1}{3},\frac{3}{2},\frac{2}{3},\frac{3}{1},\frac{1}{4},\frac{4}{3},\frac{3}{5},\frac{5}{2},\frac{2}{5},\ldots$. Amazingly, each and every reduced (simplified) fraction appears exactly once in the list. Whoa! See Chapter 21 of MATHEMATICS GALORE! (James Tanton, MAA, 2012) to learn about this sequence.

FURTHER

Of course, one can start asking about the sequences that arise from counting codes with at most three grapes per dish, or at most four per dish, and so on. Or one can count codes with no restrictions on the number of grapes per dish. Please play and explore! To see all sorts of fun ideas on this topic, check out Experience 11 of www.gdaymath.com/courses/exploding-dots/.

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Solutions appear next

SOLUTIONS

Question 1: You are left with the original sequence, but with each number increased by two.

Question 2:

a) Look at the sequence this way: **1** 11 **2** 1212 **3** 123123 **4** 12341234 **5** 123456123456 **7** ….
b) You are left with the original sequence.
c) You are left with the original sequence but with each number increased by one.

Question 3: It seems natural to guess **1** 111 **2** 121212 **3** 123123123 **4** 12341234 **5** … . Does it work?

Question 4: 512.

Question 5: $1\,|\,0\,|\,1\,|\,2\,|\,13\,|\,0$ is $32+8+2\times4+13\times2=74$ and $1\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0$ is 4096.

Question 6: Here are all the grape codes for seventeen.

1|0|0|0|1	2|0|0|1	1|2|0|1	1|1|2|1	1|1|1|3	1|1|0|5
1|0|4|1	1|0|3|3	1|0|2|5	1|0|1|7	1|0|0|9	4|0|1
3|2|1	3|1|3	3|0|5	2|4|1	2|3|3	2|2|5
2|1|7	2|0|9	1|6|1	1|5|3	1|4|5	1|3|7
1|2|9	1|1|11	1|0|13	8|1	7|3	6|5
5|7	4|9	3|11	2|13	1|15	17

Question 7: The code "17" uses seventeen grapes, the most possible. (Using the most grapes requires giving each grape the smallest possible value. Seventeen grapes in the 1s dish does this.)

The code 1|0|0|0|0|1 uses just two grapes. (And since there is no dish labeled "17" there is no code for seventeen using just one grape. Thus two grapes is the least possible count of grapes in a code for this number.)

Question 8: a) She found the only possible code for seventeen using at most one grape per dish.

Reason: There must be a grape in the 16 dish. (One grape in each of the remaining dishes adds only to $8+4+2+1=15$ -- not enough.) This then forces a single grape to be placed in the 1 dish.

Challenge: Can you prove, logically, that each number has exactly one code with at most one grape per dish?

b) Each code with 0s and 1s corresponds to adding together distinct numbers from the set …,16,8,4,2, and 1. As a sum of even numbers is even, any code that has no grape in the 1 dish represents an even sum and hence an even number. To obtain an odd sum, there must be a grape in the 1 dish.

Question 9: a) She is missing $4\,|\,0\,|\,1$. (And with patience one can check there are the only four possible codes for seventeen with at most two grapes per dish.)

b) For a sum of numbers from the set …,16,8,4,2, and 1 to be odd, there must be an odd number of 1s. As we are allowed only 0, 1, or 2 grapes in the 1s dish, there must be one grape in the 1s dish in any code for an odd number.

c) Does it seem reasonable to say that there is one code for the number zero?

d) The sequence continues: **3 5 1 5 4 7 3 8 5 7 2 7 5 …**

e) Circle every second term and find a copy of the original sequence!

1 1 2 1 3 2 3 1 4 3 5 2 5 3 5 1 5 4 7 3 8 5 7 2 7 5 …